M&T Sections: 7.4, 7.5, 7.6, 8.1

(1) Use Green's theorem with $\vec{F} = \frac{1}{2}(-y, x)$ to show that the area of an n–sided polygon D in the xy–plane with vertices $(x_1, y_1), \ldots (x_n, y_n)$ is given by

$$\text{Area}(D) = \frac{1}{2} \sum_{i=1}^{n} x_i y_{i+1} - x_{i+1} y_i,$$

where, in the above formula, we use the convention that $(x_{n+1}, y_{n+1}) = (x_1, y_1)$. This is how area is evaluated in computer graphics very efficiently, with no integration involved.

(2) Consider a torus, whose radius from the center of the hole to the center of the torus tube is a, and the radius of the tube is b. Consider therefore that $a > b$. Then the equation in Cartesian coordinates for a torus azimuthally symmetric about the z-axis is

$$(\sqrt{x^2 + y^2} - a)^2 + z^2 = b^2.$$

To derive this, one can think of the torus as a surface of revolution generated by rotating the circle $(y - a)^2 + z^2 = b^2$ around the z-axis. Find its surface area in terms of a and b.

(3) Find the flux of $\vec{F} = (yz, xz, xy)$ over the surface S which is the graph of $z = x^2 + y^2$ over the unit disk centered at the origin.

(4) Find the flux of $\vec{F}(x, y, z) = 4x\vec{i} + 4y\vec{j} + 2\vec{k}$ outward (away from the z-axis) through the surface cut from the bottom of the paraboloid $z = x^2 + y^2$ by the plane $z = 1$.

(5) Let S be the surface defined by the portion of the plane $x + y + z = -1$ satisfying $0 \leq x \leq 1$ and $0 \leq y \leq 1$ and oriented so that the normal to the plane is pointing "up". Find the flux of the vector field $\vec{F} = (x, -2y, xz)$ across the surface S.