FUNDAMENTAL OPEN PROBLEMS ON AREA PRESERVING MAPPINGS: DRAFT

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Abstract. In this note we present a number of open problems and conjectures about area presering mappings of the plane. In specific instances we will illustrate these problems via a number of well known examples: the standard famlily and the conservative Hénon family. We try to further develop a connection with renormalization and holomorphic dynamics. We believe the time is ripe for renewed vigor.

1. Introduction

Poincaré, more or less single handledly, laid bare the astonishing complexity of the dynamics of Hamiltonian systems, and saw the limits of the computational analytical point of view: to 'compute' the dynamics using for instance power series expansions. In the process of analyzing the dynamics by a geometric decomposition of phase space Poincaré discovered many of the fundamental notions of dynamics

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known today: periodic points, their eigenvalues, ellipticity and hyperbolicity, invariant manifolds, heteroclinic and homoclinic intersections, normal forms, and genericity.

Fundamental progress in the area was slow, and due primarily to the work of Birkhoff (Ergodic theorem and Fixed Point Theorem), and later through the work of Siegel, Moser, Kolmogorov, Arnol'd, Sinai, Smale, Cushman, Duistermaat, Fomenko and countless others. We choose to end the list with Conley, Zehnder, Witten, Floer and Gromov.

One of the central problems has been the issue of the Boltzmann-Gibbs ergodic hypothesis: to determine if the phase space is ergodic relative to Lebesgue measure. One of the points of KAM theory (Kolmogorov, Amsterdam 1954), is that the ergodic hypothesis is often false, since the KAM theory can 'cover' a domain of positive measure, certainly for area preserving maps, (see also for instance Arnol'd and Avez). To a large extent KAM theory provides the only obstruction to ergodicity that is understood well.

*As will be discussed later: the existence of invariant circles is **not** the only obstruction to ergodicity.*

We note as an interesting and open problem:

Problem 1.1 (Strelcyn, The "Coexistence Problem", [37])**.** *Is it possible for the phase space to divide into two **dense** sets A and B, both of positive measure, with all Lyapunov exponents zero on A while on B the maximal Lyapunov exponent is strictly positive.*

There are many of such problems, and we will discuss these in more detail below (Sea of Stochasiticity).

Physicists, primarily in Russia, for instance, Frenkel-Kontorova, (in the thirties), Chirikov (fifties and sixties), Zaslavsky, and also Taylor observed that statistical physics and confinement problems often have the dynamics of an area preserving map at its core, (or at least in a mock up version). They were able to make certain predictions about the behavior in plasma and solid state physics, based on some computational analytic understanding of the theory of area preserving maps.

The seventies and eighties brought two fundamental improvements: computing brought us a full frontal view of the complexity of the dynamics of even simple analytic examples; new, physics based, variational principles (Aubry-Mather theory) were able to transcend the analytic KAM approach and explain the well-ordered part of the dynamics of area preserving maps.

The enthusiasm of physicists and mathematicians Greene, Percival, Meiss, Collet, Eckmann, and quite a few more (Feigenbaum, MacKay, see [54], Helleman, Strelczyn, etc,) observed that the dynamics of area preserving maps is not easily explained as just a complicated mixture of hyperbolic and elliptic dynamics: the geometry is intertwined in a delicate fashion that is in its fine resolution detail very sensitive to perturbations, however in its gross characteristics always the same: stochastic, sensitive to initial conditions, numerically positive Lyapunov exponents, although rigorously out of reach. Young proved that with the addition of noise the dynamics picture dramatically simplifies and positive metric entropy is the rule. However, absent noise, to understand mathematically exactly what is going remained difficult, despite numerous attempts. From the point of view of mathematical analysis, the quest to prove serious simple statements, (for instance to solve the "problem of coexistence" or to decide if positive metric entropy prevails) about area preserving mappings became somewhat of a professional suicide mission. As far we can tell these attempts have been more or less abandoned over the last 10-20 years.

There is one exception: the theory of formal normal forms pioneered by Poincaré, Birkhoff, Arnol'd, Takens and Broer. This theory gives (some) insight in certain bifurcations prevalent in area preserving dynamics.

Investigators such as Simo have reminded the world continually of the importance of the problems with area preserving maps and of the complicated bifurcation structure seen for this dynamics.

The work of Thurston has made an enormous impact in terms of isolating a robust and fundamental mechanism at work to explain the sensitive dependence on initial conditions seen in both area preserving and dissipative maps. Thurston introduced the geometric theory of pseudo Anosov maps and train-tracks. This approach culminated in what is now known as the Pruning Front Conjecture: Czitanovic, de Carvalho and Hall, a topological approach to explain the 'creation' of horseshoes. See also Van Strien's early work (see Palis Takens 1993.)

The same computing equipment revealed that the dynamics of maps in a single variable (real or complex) is in comparison far more attractive looking and in fact essentially comprehensible. Problems that had been open and dormant since the time of Fatou and Julia became visually compelling enough for mathematicians, Sullivan and Douady-Hubbard, Lyubich, Branner, Shishikura, Devaney (see for instance Milnor Dynamics in One Complex Variable), etc, to apply conformal and quasi-conformal techniques to their resolution. At this stage, 2010, one can say that the resulting push (including Jacobson, Benedicks-Carlsson, Lyubich, etc.) has been very successful in providing an almost complete picture of the dynamics of maps of a single variable. While for a specific map it may be impossible to prove anything, we know that one dimensional families of maps typically unfold the dynamics. This approach then provides detailed answers about the prevalence of stochastic or regular behavior, with the first explained via SRB measures, and the second via essentially a group theoretic description and renormalization.

While much can still be done in one complex variable, we think that is time for the younger generation to look one dimension up and contemplate the horror that still awaits resolution.

It is our personal belief that to try to explain everything in terms of stable and unstable manifolds/structures (the Hopf picture), and weakening to non-uniform hyperbolicity, partial hyperbolicity, etc. etc, may be enough in the dissipative case where one Lyapunov exponent is non-zero (Katok, Palis-Yoccoz, Shub-Wilkinson). The area preserving case presents an alternative where Pesin theory just does not say much of anything: the case of zero Lyapunov exponents. In that case lengths can grow as a power law in time (in the very regular case) or grow, oppositely, in very irregular manner: epochs of exponential growth followed by epochs of shrinkage. We note that at the Period Doubling this growth is very controlled, and its existence thus indicates that there are possibilities to control nonlinearities. Distortion tools that are of much use in the $C^{>1}$ category are not that well developed or difficult to use (Palis-Yoccoz, Newhouse, Duarte). The C^1 category, via the work of Mañe, Bochi and Avila, is still comparatively "easy" to comprehend, though the techniques are highly non-trivial.

Theorem 1.2 (C^1 Residual Alternative)**.** *Either the map is Anosov, or the map has zero Lyapunov exponents and very regular and small exponential growth rates.*

In the smoother category, $C^{>1}$ perturbations methods fail to provide such a simple residual picture: 'return of the mess'. Furthermore the dynamics is known to to be so complicated from a topological point of view (moduli of stability) that there can be **no finite dimensional unfolding** of the dynamics near any area preserving map of interest. [1]

The core example and one that we will devote some time on is the period doubling dynamics in the area preserving case: zero Lyapunov exponents, with a very intricate and non-hyperbolic geometry. We believe that a proper understanding of this example lurks at the bottom of our ignorance.

The viewpoint developed by Bedford-Smillie, et al, comes in from the other end, via the theory of complex dynamics of more variables. Here, and in particular for the Henon family a seemingly simple and robust quasi hyperbolic picture develops that unfortunately does not see the elliptic part of the 'real' dynamics in the area preserving category.

With the older generation of mathematicians in apparent shell-shock with regards to the dynamics of area preserving maps, a younger, innocent generation, with names like Duarte, Gorodetski and Kalushin (more Russians, Ilyashenko, Gelfreich, Lazutkin,..) as well as Ecalle, having been forging fearlessly along to provide new computational analytic insights into the dynamics of area preserving maps and other examples (for instance Borel Summability).

We believe that this is an opportune moment to review the new techniques that could be also applied to this problem. We propose to formulate this in a "problem format" that is more bite-sized. We hope that by presenting enough of such problems and organizing it in a logical manner, a sort of programme can be surrected for a renewed assault on, for instance, the coexistence problem.

We do this by concentrating on three families, the Henon family, the Standard Family, and K3 dynamics. Each of these are typical in a sense besides being also holomorphic. Here is a brief description of each.

The **Conservative Henon family** (or alternatively in the deVogelaere representation) has this form:

$$(x, y) \to H_a(x, y) \equiv (a - x^2 - y, x)$$

This family is **the** analytic model for the unfolding of a homoclinic tangency in the orientation preserving case. This example can thus can be found almost anywhere in a family of area preserving maps. The dynamics of the Henon map near infinity is relatively simple, and can be decribed as a link of solenoids (Douady-Obserstforth, Buzzard).

[1] It may even be possible that the simple minded real analytic perturbation theory introduced by Broer and Tangerman, diffusion via the Heat Kernel, and the understanding of the resulting perturbed, but real analytic (often in fact "entire") dynamics may be a key tool in this respect. Heat Kernel methods were initially seen as useful in index theory.

The **Standard family** (see survey by Meiss and Wikipedia entry) is defined on the cylinder $\mathbb{S}^1 \times \mathbb{R}$:

$$(x, y) \to S_k(x, y) \equiv (x + y + \frac{k}{2\pi} sin(2\pi x), y + \frac{k}{2\pi} sin(2\pi x))$$

This family is prototypical in that it incorporates periodic forcing, and tends to mix elliptic, hyperbolic and well-ordered behavior, that is to a large understood via Aubry-Mather theory. (One can replace the sinusoid by almost any other non-constant periodic function.) The view from infinity of a map in the standard family, certainly when viewed in two complex dimensions, is not simple. Its dynamics is, by analogy, more like an entire function than a polynomial.

The **K3 family** is a good prototype to learn the off-real dynamics, combining holomorphic pseudo Anosov behavior (postive entropy) with elliptic behavior (Siegel disks). The real dynamics of the K3 family is like every other area preserving map. (Different from the Trace map). The K3 family is algebraic and defined for this discourse on a bi-quadratic surface $K3(A, B)$, with A and B fixed:

$$(x^2 + 1)(y^2 + 1)(z^2 + 1) + Axyz = B$$

and defined algebraically as follows. Let (x, y, z) be a point in $K3(A, B)$. Since the equation is quadratic there is a unique point $\iota_X(x, y, z) = (x', y, z)$ in the same surface. ι_X is thus an involution. Similarly define involutions $\iota_Y(x, y, z) = (x, y', z)$ and $\iota_Z(x, y, z) = (x, y, z')$. The composition of any two, for instance $\iota_X \circ \iota_Y$, preserves a foliation, (in this case $z = const$) and are thus integrable. The $K3$ map is the following composition: $\iota_Z \circ \iota_Y \circ \iota_X$. Each of these maps preserves the level surface of the function $(x^2 + 1)(y^2 + 1)(z^2 + 1) + Axyz$. Since each of these maps is obviously volume preserving, the induced map on the $K3(A, B)$ surface preserves, by Liouville's theorem, an induced area form. We refer to the work of Cantat and McMullen for further information as well as a major tool for proving hyperbolicity/Anosov behavior: the Kähler cone. Of particular interest is the fact that K3 have minimal entropy in their class, a property that is analogous to pseudo-Anosov maps, while also permitting the simultaneous existence of Siegel disks. This situation is therefore rather different from hyperbolic toral automorphisms that are then automatically Anosov.

It is the interaction between these three examples that, we hope, can elucidate what is really going on in even one example of interest.

A Word Of Warning: If we learned anything about the theory of one variable, it is that the mathematics becomes a lot easier when looked at from the holomorphic perspective. However the holomorphic theory (Douady-Hubbard etc.) needed to be almost fully developed before one could return to the 'real' case, the smooth category. The smooth category is seemingly too flexible to comprehend easily: nonlinearity tools are still sorely lacking in that category (Comment here: the main trick is to prove suitable convergence to the real analytic, and thus holomorphic case). The real analytic/holomorphic assumption introduces a notion of algebraic **rigidity** that seems to be indispensible for rapid progress.

We will also see this in the two dimensional area preserving case. At some point one may need to take a holomorphic, or at least real analytic point of view, and use it advantageously. It is through the work of Bedford-Smillie-Lyubich, Hubbard, Sibony, Fornaess, Buzzard and DuJardin that we think progress may be possible in our life time.

What is particularly lacking is a good tool to visualize anything in two complex variables. Some computational tools have been developed by Hubbard and Ishii(?) et al.

How do algebraic geometers do 'it'?

Remarks:

(1) This list is not meant to be a comprehensive overview of the literature, but rather my view on what is perhaps achievable.
(2) Problems may be posed in a number of different ways. When stated as "Conjecture" I indicate that there is more universal support for the statement. The word "Problem" has two meanings: either the issue is of a more open-ended and of a vaguer nature, or oppositely, technically doable, (as in a calculus type problem).
(3) While it would be ideal to have the best references listed, the field is fairly vast. Instead we opt as default reference mechanism: to provide enough information so that the search string "Hénon ref", for instance "Hénon Devaney-Nitecki" will yield relevant references via any search engine".

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2. More on the Henon Family

The purpose of this note was initially to only state a number of open problems regarding the conservative Hénon case , i.e. the map:

$$H_a(x, y) = (a - x^2 - y, x)$$

The parameter a varies in the interval $[-1, 5.69995117187506] \sim [-1, 5.7]$. For $a < -1$ the real dynamics is trivial (the complex is not). For $a > 5.7$ (or thereabouts) the dynamics is uniformly hyperbolic, and the nonwandering set (also in the complex) is real, hyperbolic, at entropy $log(2)$, and the dynamics is conjugate to the full 2-shift.(Devaney-Nitecki, Bedford-Smillie (I-XII), Newhouse)

Remarks:

- The map H_a is a subclass of the following family:

$$H_{a,b}(x, y) = (a - x^2 - by, x)$$

  which has Jacobian equal to b. The dissipative case corresponds to $|b| < 1$. The notation H_a will refer to the case $b = 1$.
- With the identification $(x, y) = (x_n, x_{n-1})$ it is sometimes useful to represent the Hénon map in the following form:

$$x_{n+1} + x_{n-1} = a - x_n^2$$

which shows that the map is reversible in a simple manner.

- Reversibility for the map H_a can be expressed in the following manner also: Let $R(x,y) = (y,x)$ denote reflection in the diagonal, then

$$H_a \circ R = R \circ H_a^{-1}$$

When the context is clear (real or complex) we will use these standard notations: K^+, respectively K^-, is the set of points with orbit that is bounded for positive, respectively negative, times. $K = K^+ \cap K^-$. Since the map is reversible $K^- = R(K^+)$. The boundary of K^+ is J^+, while the boundary of K^- is called J^-, again, $J^- = R(J^+)$. Finally $J = J^+ \cap J^-$. When the context (real or complex) for a set A defined as above is not clear we will say $A_{\mathbb{C}}$ when referring to the complex context or $A_{\mathbb{R}}$ its restriction to the reals.

While there are no attractors or repellors in the conservative Henon family, it is possible for K to have non-empty interior. Components of its interior are of a special type: rotation/Reinhardt domains (see Bedford-Smillie II) on which the dynamics is entirely classified as Siegel Domains (there is confusion in the language) or Herman Domains. These domains may be thought of as foliated by Siegel disks or by Herman rings. Siegel domains do occur in the Hénon family. Such rotation domains do not intersect the reals, for a simple reason: on the reals the non-resonance condition fails.

Problem 2.1. *Siegel Domains ought to be plentiful in the area preserving Hénon family. Is this the corresponding Newhouse phenomenon in the holomorphic area preserving case? I.e. do tangencies 'create' Siegel domains?*

In the theory of one complex variable, Herman rings do not exist for polynomial maps. They do exist for rational and entire maps.

Problem 2.2. *Do Herman* ***Domains*** *exist in the Hénon family?*

3. Meta Problems

We also note the following meta problems: Tangencies, etc, create also moduli of stability, making equivalence classes, "modulo conjugacy" too small to be useful. The Teichmuller classification "modulo conjugacy up to isotopy relative a finite set" yields equivalence classes that are too big, but usually contain pseudo-Anosov maps that 'explain' the dynamics, but usually on a set of measure of zero. Gardiner and Lakic (Quasiconformal Teichmüller theory), expanded Teichmüller Theory relative a finite set, to a notion of "asymptotically conformal", which behaves in many ways correctly to be useful for Teichmuller theory relative to an infinite set, for instance a Cantor set. As far as I know the notion of "asymptotic conformality" has not yet been very successful in holomorphic dynamics of one variable, or real dynamics in two variables. There is a formulation of the Period Doubling Renormalization operator relative to a Period Doubling Cantor set.

Problem 3.1. *Find a use for the notion of* ***asymptotically quasiconformal*** *in two dimensional dynamics. Under what conditions can Thurston's theory be expanded into this realm?*

Remark 3.2. There is additional quasi conformal and affine structure (Buzzard Verma) on certain 'instrinsic' (measured) laminations, see also further sections.

Problem 3.3. *Can the Gardiner theory be expanded relative to for instance laminations? How about relative to a measured foliation induced by a pseudo-Anosov?*

Remark 3.4. These problems are vaguely stated, but maybe accessible with some further thought. The analysis is non-trivial, but maybe most of the work is **already done**, and some attention is needed from the harmonic analysts.

Problem 3.5 (to Fred Gardiner:)**.** *Is "Asymptotically conformality" on a set E equivalent, in some moral sense, to $C^{1+\alpha}$ restricted to a E????? If not, is there then a better solution? For instance with respect to a product structure?*

Problem 3.6. *Develop the theory of* ***asymptotically affine*** *in two dimensional dynamics. Under what conditions can Thurston's theory be expanded into this realm?*

4. Quick Tour through Parameter Space

When $a = -1$ there is a single parabolic fixed point, with a center stable and a center unstable curve. The complex dynamics on the center stable and the center unstable complex curves is also known, the regularity of these curves is understood, with dynamics conjugate to $z \to z + 1$([14], following the methods of Ecalle and Lazutkin).

Problem 4.1. *Assume that $a = -1$:*

(1) *Describe the complex dynamics.*
(2) *Are K and J connected?*

Remark 4.2. When $a = -1$ one obtains with a slight change of coordinates the 'Mother' of nonlinear recursion relations common in area preserving maps: $x_{n+1} + x_{n-1} - 2x_n = -x_n^2$. It would seem surprising that it would be unknown if the Julia set were connected or not.

When a is slightly larger than -1 There are a number of things that happen. The parabolic fixed point is replaced by a pair of fixed points, one with positive eigenvalues β_+, immediately hyperbolic and one with negative eigenvalues β_- which for a in the range $(-1, 3)$ is **elliptic**. When $a > -1$ the stable and unstable manifolds of β_+, $W^s(\beta_+)$ and $W^u(\beta_+)$, intersect for all $a > -1$ (Devaney-Nitecki) and due to the reversibility of the Hénon map and simple geometric insights into how the map **turns**. The entropy of the map is positive when $a > -1$. It was earlier conjectured by Milnor (not clear how seriously though) that the entropy is monotone in the parameters.

Problem 4.3. *Is the entropy of the Hénon map on the reals monotonically increasing with a in the conservative case?*

Meiss et al,[29], and Yorke et al ("Non monotonic") found that the bifurcation theory is **not** monotone in a.

In the eighties, (Alligood-Yorke [30, 31]) proved in both the dissipative case as well as the conservative case that, by analytically tracking the bifurcations between the simple dynamics $a < -1$ and the hyperbolic dynamics $a > 6$ that there are **complete** period doubling sequences. As a result one has the analog of the **existence** of a map in the quadratic family with the dynamics of an adding machine in the base 2 (a.k.a. as period doubling, or Feigenbaum dynamics). We state this as a theorem, as this result is not generally known.

Theorem 4.4. ***Alligood-Yorke*** *There are maps in the conservative Hénon family that have a Feigenbaum like invariant Cantor set, on which the dynamics is conjugate to the adding machine. ('Feigenbaum dynamics').*

Remark 4.5.

- The bifurcation tree for the conservative Hénon map was analyzed extensively also by MacKay ([8]). He also noticed that via saddle node bifurcation certain periodic orbits just 'have to' appear, out of the 'complex blue', This happens for example for period 5 where some are created by a saddle-centre bifurcation.
- In the dissipative Hénon map the period doubling sequence is apparently known to be not monotone (Meiss Hénon Non Monotonic).
- MacKay and certainly others also observed that the period doubling appears to be monotone in the parameter a.

Problem 4.6. *Prove that the period doubling sequence is monotone in the conservative case*

If necessary we will assume that there is a unique parameter value a_{2^∞} (and map H_{2^∞}) in the conservative Hénon family with Feigenbaum dynamics.

ADD MORE There are many papers describing aspects of the bifurcations with increasing parameter for the Henon family, FIND SOME...

5. Sea of Stochasticity

(Duarte Abundance of Elliptic Isles) has made a very precise analysis of the homoclinic tangle produced by the intersections of stable and unstable manifolds of β_+ for a close to -1, in an interval $[-1, -1+\epsilon)$.

Theorem 5.1 (Duarte)**.** *This example contains the conservative Newhouse phenomenon: there are many parameter values in the interval $[-1, \epsilon)$ where the map H_a has many elliptic points, that occur near parameter values and locations in the plane where H_a has a homoclinic tangency. The parameter value $a = -1$ is a Lebesgue point of density of such parameters.*

Since the Hénon map is a normal form for a homoclinic tangency, Theorem 5.1, suggests that the Newhouse phenomenon is indeed widespread in conservative systems:

Conjecture 5.2 (Check if this has been proven already, Gorodetski was not too precise.)**.** *There is a (large) open set in the C^k-topology, ($k > 1$) of area preserving maps where the following property is residual: Elliptic Periodic points and thus elliptic islands are dense in the non-wandering set.*

Theorem 5.3 (Gorodetski)**.** *Elliptic islands are prevalent in the Standard Family S_k: For $k > 10$ there is a residual set of parameters for which the closure E_k of the set of elliptic points is a transitive invariant set of Hausdorff dimension equal to 2, and E_k is $\dfrac{1}{k}$ dense.*

Recall the dynamics of Hénon like maps: most points escape to infinity under forward iteration. Clearly points that are trapped in elliptic islands do not escape to infinity.

Problem 5.4 (Area Preserving Henon)**.** *Is the Lebesgue measure of the non wandering set $K_{\mathbb{R}}$ equal to the Lebesgue measure of the union of the elliptic islands?*

Remark 5.5. The C^1 (and the C^0) topology behaves rather differently in the conservative case. C^1 perturbation theory in the conservative case (Mane, Avila) is rather wild: Avila has a good C^1 perturbation method to show in the conservative category that there is a dichotomy: either the map is Anosov or the map is a C^1 limit of maps with only zero Lyapunov exponents. Furthermore more the dynamics associated with the zero Lyapunov exponents is regular in the sense of (Avila-Bochi).

Problem 5.6. *What bifurcations in the topology of K are associated with homoclinic tangencies (real or complex)?*

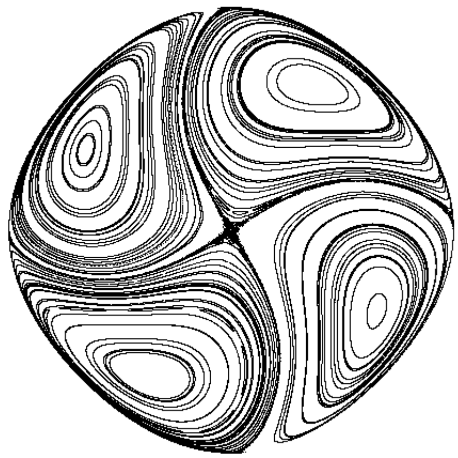

FIGURE 1. Orbits for K3, the tame case. (Image thanks to McMullen).

McMullen ([32]) suggested the following common sense conjectures about the Sea of Stochasticity.These conjecture provide a framework that is analogous to the Palis Conjecture for dissipative systems.

These conjectures are born out of observations from a limited but varied set of examples: algebraic, real analytic, and smooth. These observations were made initially by MacKay ([8]), and can bear some precision. While it may be difficult to make any statement about any map in particular, the philosophy pioneered by Jacobson, and further applied by Benedicks-Carleson, etc. should make it possible to make certain statements about suitably transverse families (to be defined). Algebraic, and real analytic examples **appear** to be rigid enough to be transverse, but so is C^{333} and likely C^3. We note that there is a general method to imbed a map into a family, diffusion via convolution by the heat kernel. This perturbation method tends to preserve many properties ([41]) and allows one to deduce Kupka-Smale properties in general real analytic categories.

Let $F_t : X_t \to X_t$ be a suitably transverse (say, one-dimensional) set of **real** area preserving maps, through the map F_0 which we assume to be **not** Anosov.

Conjecture 5.7. ***SOS 1:*** *There exists and open and dense set of parameters t for which there* ***exist*** *elliptic islands.*

Conjecture 5.8. ***SOS 2:*** *There exists and open and dense set of parameters t for which elliptic islands are dense.*

Conjecture 5.9. ***SOS 3:*** *There is* ***always*** *an ergodic component of positive measure. (See Figures 2 and 3) (The largest Lyapunov exponent may or may not be positive)*

Conjecture 5.10. ***SOS 4:*** *There exists a set of parameters t of* ***positive measure*** *for which there are* ***no*** *elliptic islands.* [33, 34]

Conjecture 5.11. ***SOS 4':*** *If F_0 is special, for example the standard map with k large enough, and an isotopy F_t, then there exists a set of parameters t of* ***positive measure*** *for which F_t is ergodic.*

We refer to the Strelcyn [37] reference for the precise statement of the "coexistence problem" (see also the introduction).

Conjecture 5.12. ***SOS 5:*** *There are area preserving examples that satisify Strelcyn's coexistence condition/*

In the Thurston-Nielsen classification of surface homeomorphisms up to isotopy there is always a dichotomy the map is reducible (leaves invariant a multi curve) or is irreducible (periodic case or pseudo Anosov case).

Conjecture 5.13 (Turaev;)**.** *There is a real analytic map defined on an annulus, bounded by invariant curves that is infinitely reducible. This is the standard picture of islands within islands.*

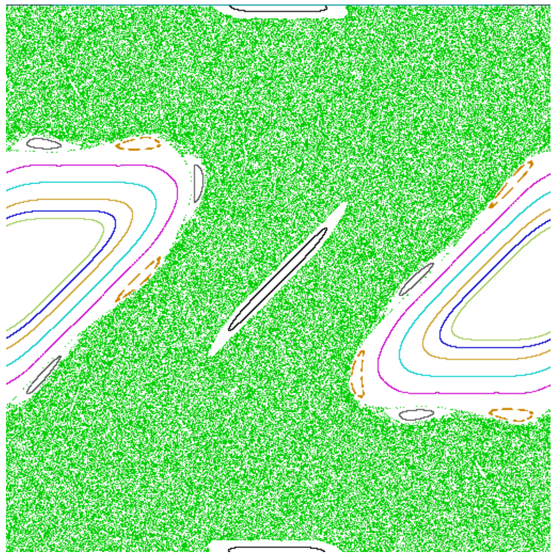

FIGURE 2. Standard map K=2.0;

Problem 5.14 (Bonk, Kleiner, Merenkov, Rigidity of Schottky Sets)**.** *Are Schottky sets (complements of at least 3 spheres) an appropriate model for ergodic connected components of positive or zero Lebesgue measure?*

ADD MORE HERE

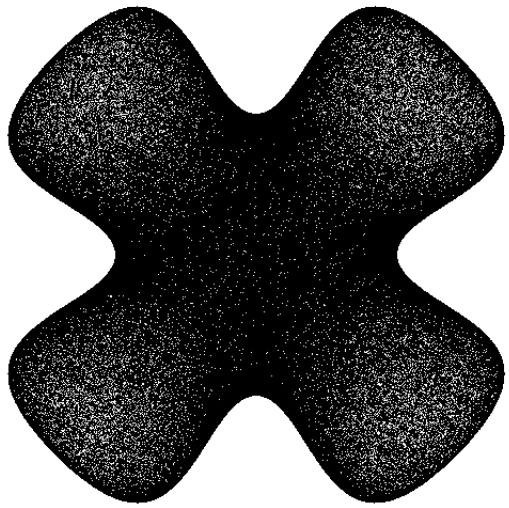

FIGURE 3. Obits for K3, the seemingly ergodic case. (Image thanks to McMullen)

6. Big Invariant Circles

The bifurcation from $a = -1$ is of Neimark-Sacker type[2]. This bifurcation leads to the the creation of invariant circles in the reals that emanate from the elliptic fixed point. The rotation number of such an invariant circle is determined at the moment of its creation: the argument of the eigenvalues at the elliptic fixed point β_- and therefore increases monotonically.

[2]This bifurcation is the analog of the Takens-Bogdanov bifurcation for flows and was sometimes called secondary Hopf bifurcation

When $a = 3$ the point β_- has a period 2 bifurcation: β_- becomes hyperbolic, and in addition there is a new real period two elliptic cycle. This period two cycle exists also for $a < 3$ but is then a hyperbolic cycle off the reals. This is a general phenomenon: a **complex conjugate** hyperbolic pair becomes a real elliptic pair, during period doubling.

When $a = 3$ there are still invariant circles surrounding β_-. These disappear when a is around 4. We make a numerical observation: these invariant circles, as long as they exists appear to be **graphs** in polar coordinates, centered at the point β_-. It would be interesting if one could prove this with a simple argument. [3]

abcd

Problem 6.1 (Maybe John Franks knows this.)**.**

- *Is there an action-angle coordinate system centered at the point β_- relative to which the map has the monotone twist property, when this point is elliptic?*
- *How far does this system extend?*
- *Is the Hénon map a monotone twist map in that coordinate system?*

Problem 6.2. *For the Hénon map find a simple (Matherish) argument to show that there are no invariant circles surrounding β_- when $a > 4$.*

Clearly a heteroclinic intersection between stable manifolds of β_- and unstable manifolds of β_+ implies that there are no invariant circles surrounding β_-. [4] The following conjecture is the converse:

Conjecture 6.3. *When there are no invariant circles surrounding the point β_- then the stable manifold of β_- intersects the unstable manifold of β_+.*

For twist maps on the annulus there is an analogous converse: if there are no topologically invariant circles then there are orbits with $\alpha-$ ω-limit sets on both boundaries (if not: then there would be an invariant circle by construction.) We

[3] We briefly review the corresponding situation for the standard map S_k: $(x, y) \to (x + y + \frac{k}{2\pi} sin(2\pi(x)), y + \frac{k}{2\pi} sin(2\pi(x)))$. In these coordiantes x is periodic of period 1. The standard map S_k is defined on the cylinder $\mathbb{S}^1 \times \mathbb{R}$. Mather, [42] proved, using a very simple argument that for $k > \frac{4}{3}$, there a no topologically non-trivial invariant circles for the standard map S_k. This method is very robust, *i.e.* it can be applied to many variants of the standard map (change the sinusoid to another periodic function). This method can be used to produce an a priori bound on the non-existence of such invariant curves for a broad class of maps. MacKay ([8]) using computing and Mather's cirterion, proved that when $k > 0.971635. \sim \frac{63}{64}$ there are no topologically non-trivial invariant circles of any rotation number.

[4] Since Hénon maps are reversible, an intersection between $W^s(\beta_-)$ and $W^u(\beta_+)$ implies an intersection between $W^s(\beta_+)$ and $W^u(\beta_-)$.

note that it rather straightforwardly to prove using numerics that such manifolds intersect when they do so transversally. It is not easy to so geometrically.

Problem 6.4. *Find a geometric argument based on simple intuition about the Hénon map, that when 'something', the nonlinearity, say measured by 'a', is sufficiently large, that there is then a heteroclinic intersection between $W^s(\beta_+)$ and $W^u(\beta_-)$.*

Remark 6.5. Since invariant circles persist while the point β_- becomes hyperbolic (i.e. for a range of parameters beyond $a = 3$) this can not be too obvious.

These problems are related to the following observation made by MacKay ([8]. Using a high resolution grid of initial points, MacKay measured the Lebesgue measure $\lambda(a)$ of the non wandering set of H_a. [5]

- Whenever there is period three bifurcation $\lambda(a)$ **decreases sharply**.
- When $a = a_{2^\infty}$ then $\lambda(a)$ appears to be equal to zero.

These measurements could easily be flawed by many small elliptic islands ('see of stochasticity') that were missed in the counting.

Problem 6.6. *Has this experiment been repeated at very high resolution studies?*

MacKay did carry out scaling analysis for $\lambda(a_{2^n})$ as $n \to \infty$.

Conjecture 6.7. *Small elliptic islands exist at the period doubling parameter a_{2^∞} at all length scales.*

Remark 6.8. Relatively large elliptic islands have not been observed at the period doubling parameter, and the general conjecture that the elliptic island have small density.

Conjecture 6.9. *At the period doubling parameter $a = a_{2^\infty}$ the set of escaping points has **positive** Lebesgue density on the invariant Cantor set.*

Conjecture 6.10. *For any map F in the stable manifold of the period doubling operator the set of escaping points (needs to be properly defined) has **positive** Lebesgue density on the invariant Cantor set C_F.*

[5]When a is outside the interval $[-1, 6]$ $\lambda(a) = 0$: the map has empty non wandering set or the non wandering is hyperbolic.

WHY IS THERE NO SUBSECTION HERE?

If we now consider the complex Hénon maps with the same parameters it would be very useful to be able to trace topological changes in $K(a)$, resp $K^{\pm}(a)$,with increasing a, by looking at the dynamics on the reals. Let $K(a, x)$ denote the connected component of $K(a)$ containing the point x. Clearly $K_{\mathbb{R}}(a, \beta_+)$ and $K_{\mathbb{R}}(a, \beta_-)$ are disjoint when $-1 < a < 3$ (or as long as there is an elliptic island surrounding β_-).

Problem 6.11. *For $-1 \leq a \leq 3$ (or as long as there is an elliptic island surrounding β_-:*

- *Are $K_{\mathbb{C}}(a, \beta_+)$ and $K_{\mathbb{C}}(a, \beta_-)$ disjoint?*
- *Are there any topological changes in $K_{\mathbb{C}}(a, \beta_-)$*

Problem 6.12. *Describe the change in topology between $K_{\mathbb{C}}(3, \beta_-)$ (when there are invariant circles surrounding β_-) and $K_{\mathbb{C}}(4, \beta_-)$, when there are no such invariant circles surrounding β_-.*

7. Pruning Front Questions

Here we summarize a bit as to what is known about pruning fronts for the Hénon family, and what might be a good conjecture.

ADD More, also ask Andre about this.

In particular there is an question of Lyubich: can we 'approximate' the period doubling map via Pseudo Anosovs. The advantage of the latter is that they are also (in some sense) area preserving.

Problem 7.1. *Is the area preserving or the reversible category* ***special*** *in the pruning front perspective?*

8. Period Doubling Dynamics

We first begin with a summary of the results to date regarding Period Doubling in the conservative case.

Theorem 8.1. ***Eckmann-Koch-Wittwer*** *There exists a real analytic area preserving fixed point of renormalization.*

Some of the properties of this fixed point have been determined rigorously by numerical means (see Eckmann-Koch-Wittwer). Work is underway (Gaidashev et al) to use softer methods to determine the existence of this fixed point: numerical apriori bounds, and Schauder Fixed Point Theorem.

Specifically:

Conjecture 8.2. ***Gaidashev-Koch is working on this type of statement*** *The fixed point F of renormalization is* ***close*** *to a Hénon map.*

Remark 8.3. In the setup of Eckmann-Wittwer-Koch the map is defined by an action S. Reversibility is expressed as a symmetry with respect to the reflection $(x, y) \to (x, -y)$. The fixed point map F is not close to $H_{a_{2^\infty}}$, but close to a conjugate of it, the corresponding deVogelaere map.

According to Gaidashev (private communication), while the unstable eigenvalue is known at 9 digits of accuracy, the best estimate of the spectral radius along the stable subspace of the renormalization operator at the fixed point F is less accurate and roughly 0.85

The dynamics of this fixed point was analyzed by Gaidashev et al. We are using the same notation for the fixed points β_+ and β_-.

Theorem 8.4 (Gaidashev and Johnson, [43, 44])**.**
(1) *F has a transverse heteroclinic intersection between the stable manifold of β_- and the unstable manifold of β_+ (and vice versa via reversibility).*
(2) *F has a period doubling Cantor Set.*
(3) *If G is on the stable manifold of the renormalization operator and if the renormalizations of G converge sufficiently rapidly to F (i.e. at a prescribed upper bound $r \sim (0.002)$ on the rate of convergence), then the conjugacy on their respective Feigenbaum Cantor sets is Bi-Lipschitz.*

The following conjecture is very reasonable (Lipschitz implies higer smoothness).

Conjecture 8.5. *If G is on the stable manifold of the renormalization operator and if the renormalizations of G converge sufficiently rapidly to F (i.e. at a prescribed upper bound r on the rate of convergence), then the conjugacy on their respective Feigenbaum Cantor sets is $C^{1+\alpha}$.*

.

The bound for r has a geometrical interpretation, see [43] and seems to be almost optimal, if one disregards a peculiar geometry that is inherent in the area preserving case. This geometry was also not used in Gaidashev's proof. We conjecture therefore, also on esthetic grounds:

Conjecture 8.6. *All maps in the stable manifold of the renormalization operator are on the period doubling Cantor set Bi-Lipschitz (and thus $C^{1+\alpha}$) conjugate.*

Problem 8.7. *What is the appropriate class of scaling functions to describe the intrinsic geometry of the period doubling Cantor Set?*

Remark 8.8. These scaling functions should take not just length, but also orientation change effects into account. Note that the areas of the pieces at depth n are the **same**, and that every piece has a corner that has dynamic signficance (orbit of period 2^{n-1}). So one approach might be to use to points, and an oriented frame along the axes, in such a way that the areas are correct and consistent with the $SL(2m\mathbb{R}$ action of the derivative.

While the dynamics of H_{a^∞} on the Period doubling Cantor C is in a sense obvious (adding machine σ on the cocompact group $2^{\mathbb{N}}$) we do not understand its embedding in the reals too well.

Here is a simple question:

Problem 8.9.

(1) *What are the limits of the sequence of iterates of H_{a^∞} on C in the C^0-topology?*
(2) *How about the C^α-topology $0 < \alpha < 1$?*
(3) *How about the C^1-topology?*

The next section puts the imbedding properties in more perspective.

9. Cocycles for the Renormalization Fixed Point

The following theorem is an extension of the corresponding theorem due to Carvalho-Lyubich-Martens [16]:

Theorem 9.1 (Tangerman [38])**.** *If H is conservative and reversible and H has a invariant Cantor set C_H with the period doubling dynamics, then there is no* ***continuous*** *invariant line field on C.*

´ The argument goes by contradiction, as in [16]. Of interest however is the ergodic theory of the underlying $SL(2,\mathbb{R})$ cocycle, when the underlying dynamics is topologically minimal (the adding machine in this case or irrational rotations studied by Avila).

Consider the projectivized action PH of the derivative of H on directions and ellipses, (think Beltrami differentials) define on $C_H \times \mathbb{D}$, where $\mathbb{D}$ is the unit disk and defined as:

$$(x, v) \to (H(x), PDH_x(v)$$

Here PDH_x is the Mobius transformation associated to the derivative DH of H that maps directions to the unit circle. If H is topologically minimal then the ergodic theory of PH should not be that complicated. Avila analyzed the case for irrational rotations. [6] In a typical situation these Móbius maps themselve may be elliptic or hyperbolic or a parabolic. (There is here a more general context Frenkel-Kontorova, Matthieu, Fibonacci) Extend this analysis to the adding machine on $2^{\mathbb{N}}$, or similar skew-products for other 'cocompact' groups. Any invariant measure on $C_H \times \mathbb{D}$ disintegrates to a measure on each of the fibers $\mathbb{D}$, equivariant by the dynamics on the base. The base has a unique invariant measure μ_H and one can talk about properties of the disintegrated measure a.e relative to μ_H.

Problem 9.2. *Understand the general ergodic theory of such actions.*

[6] Yoccoz has analyzed the case when the map H is defined on a Cantor set for the shift map (more complicated case in some sense) and the map $x \to PDH_x$ is piece wise constant.

For an **ergodic** invariant measure, the options are limited, see also (See Yoccoz WHAT REF?, Avila WHAT REF?):

(1) *Rotation Case:* The support of the measure is μ_H a.e bounded in the Poincare metric on the open unit disk. The hyperbolic barycenter is equivariant under the dynamics. By conjugating this barycenter fiberwise to be at the origin of the $\mathbb{D}$ one concludes that these maps are conjugate to rotations.

Problem 9.3. *Under what conditions may one bootstrap the* ***almost everywhere*** *to everywhere? in the rotation case?*

(2) *Not Rotation Case* In this case the support of the ergodic measure is contained in the boundary $\mathbb{S}^1$ of $\mathbb{D}$. Here the notion of conformal barycenter due to Douady-Earle is useful, since it is equivariant by the dynamics on the base. There is only one case where the conformal barycenter is not defined: two points on the boundary with equal weight $(= \frac{1}{2})$. In that case there an invariant **pair** of directions (a.e.).

Thus this case has very few possibilities.

(a) *One Invariant Direction:* the support consists of a single point on the boundary. In which case there is an invariant direction a.e. Thius could happen in a hyperbolic situation, a degeneration (say 1 positive Lyapunov exponent, as in the dissipative case [16].

(b) *An Invariant Direction Pair:* The support of the disintegrated measure on the fiber consists of two points with equal weight (a.e). DISCUSS more.

(c) *Rotation Cases:* The conformal barycenter is contained in the open unit disk. In which case we should be back to the rotation case.

Conjecture 9.4.

(1) *PH_{2^∞} is uniquely ergodic.*
(2) *and has a unique (a.e) invariant direction.*

This should imply the following:

Conjecture 9.5. *The invariant Cantor set for H_{2^∞} is rectifiable.*

Let p_n be the sequence of periodic orbits of period 2^n that converge to the Cantorset. This periodic orbit is hyperbolic. On such a periodic orbit there are two direction fields, corresponding to stable and unstable directions: $E^s(p_n)$ and $E^u(p_n)$.

Problem 9.6.

(1) *In what sense do $E^s(p_n)$ and $E^u(p_n)$ converge?*
(2) *In what sense do they have the same limit?*

One can also consider not just limiting directions but also the limits of $W^s(p_n)$ and $W^u(p_n)$. In the dissipative case it is possible to argue that their limits, at least generically, form a laminations in a neighborhood of the invariant Cantor set. Exceptions are caused by homoclinic/heteroclinic tangencies. While I may believe that such homoclinic tangencies do not exist in the conservative Hénon case at the period doubling point, that would be a rather unique situation that would need explanation (Of course there is only one such map in the Hénon family). (Also near such tangencies elliptic islands should be found if we enlarge the space of maps to be conservative Hénon like see below).

Problem 9.7.

(1) *Is there any reason to believe that the limits of $W^{u/s}(p_n)$ exist?*
(2) *If so, is the limit a lamination?*

10. Renormalization Operator: Real

The first issue to be explained is what was remarked in MacKay([8]): numerically the geometric scalings at the fixed point of renormalization are as follows: $\lambda \sim \pm\frac{1}{4}$, $\mu \sim \frac{1}{16}$ and so they are **nearly but not exactly resonant**: $\frac{\lambda^2}{\mu} \sim \pm 1.03$.

Problem 10.1. *Why are these geometric scalings nearly resonant? (see below for an explanation)*

Problem 10.2. *What is the relevant category: reversible? conservative?*

Let's assume that F is a reversible conservative map, close to the Hénon family. Also assume that F has the following Hénon like normal form:

$$(x, y) \to (f(x, y), x)$$

Consider the maps FR and R. Both are involutions, area preserving and orientation reversing. FR and R do not commute: $F = FRR$, while $F^{-1} = RFR$ (reversibility). While R is a linear reflection in the diagonal, FR is definitely nonlinear. Denote the axis of symmetry of FR as $Fix(FR)$, the set of points that are fixed by FR. $Fix(FR)$ is a parabola $x = Q(y)$, and FR preserves the horizontal leaves.

Problem 10.3. *Consider the group on two generators generated by FR and R. What additional structure is there that could be exploited (besides lines of symmetry)?*

Let $\mathbb{V}$ be the vertical foliation, and let $\mathbb{H}$ be the horizontal foliation. Then $F(\mathbb{V}) = FRR(\mathbb{V}) = FR(\mathbb{H}) = \mathbb{H}$. The foliation $F(\mathbb{H}) = FR\mathbb{V}$ consists of 'parabolas'. Its leaves are of the form $x = Q_2(y)$ with Q_2 (close to) quadratic and have second

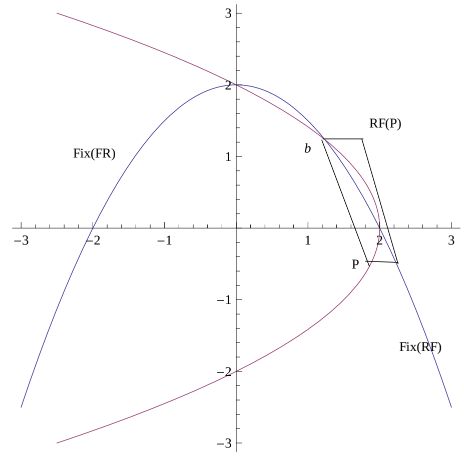

FIGURE 1. Sketch of the Box Construction. Shown are the parabolas $Fix(FR)$, $Fix(RF)$, the corner β_-, labeled 'b' in the picture, and the points P and $RF(P)$.

derivative that is about two times that of the curve describing the axis of symmetry of FR.

Next consider $F^{-1}(\mathbb{V})$. Since $F^{-1} = RFR$ we then obtain that $F^{-1}(\mathbb{V}) = RF(\mathbb{H})$. Its leaves are thus the reflection of the leaves of $F(\mathbb{H})$ and are parabolas $y = Q_2(x)$.

Now note that $RF = RFRR$ itself is a nonlinear reflection that preserves the vertical foliation, and is a vertical reflection in the parabola $y = Q(x)$.

Assume that F has a fixed point β_- and assume that it is hyperbolic, i.e. of flip type, i.e. has negative eigenvalues.

We now define its **geometric renormalization**, using a process that is almost the same as that defined in [16]). It is defined in an intuitive manner, but is based on preserving the reflective properties that are needed.

We construct a geometric box B, see Figure 1, that has the point β_- as a corner (on its top to the left) [7]

(1) Consider the horizontal line segment l to the **right** of β_-
(2) Consider its reflection $RF(l)$ and determine its first intersection P (beyond the point β_-) with the curve $Fix(FR)$: below and to the right of β_-. The resulting line segment $[\beta_-, P]$ on $RF(l)$ is the **left** side of the box. The point $RF(P)$ is of course on l again. The horizontal line segment $[\beta_-, RF(P)$ forms the **top** of the box.
(3) The **bottom** of the box B is constructed as follows. Draw the horizontal from the point P to the parabola $Fix(RF)$, to the right of P. This line segment forms the bottom of the box.
(4) The **right** side of the box is obtained using the reflection RF of the bottom just constructed.
(5) This produces **an almost affine** box B.

[7]We note that a similar region can also be defined using stable and unstable manifolds of β_-.

(6) The fixed point set of the involution RF is a 'diagonal' in the box B, and is almost a straight line.
(7) The fixed point set of the involution FR is a parabolic like curve $x = Q(y)$ that connects β_- with the point P.
(8) $FixFR$ intersects the diagonal of the box $Fix(RF)$ in a second point (besides the corner point β_-. This second point is a point β_1 of period 2.
(9) These are the alternatives:
    (a) Too far: This segment on $Fix(FR)$ is not entirely contained in the box B. In that case we have gone too far and a horseshoe has been created.
    (b) Too shallow: The segment is almost up against the right side. In that case the point β_1 is elliptic.
    (c) Just deep enough: The point β_1 is hyperbolic, but of flip type for the map F^2 and we can renormalize again.

    Also notice that the corner β_- is an orientation preserving fixed point for the map F^2 restricted to B.

The "near resonance" statement, see the statement above, is really a statement abouth the ratio of the sides of this box. A simple calculation, near the Hénon family, shows that the ratio of width to height has to be close to 1:4. An explanation for this ratio is that the segment on the fixed point set of the involution FR should be contained in this box for the map to be again renormalizable. Since this segment has a **quadratic** singularity inside the box, the ratio of height to width must approximately be in the 1:4 ratio.

Conjecture 10.4. ***Apriori bounds (naive version):*** *Let F be a conservative reversible map near the Hénon family that is once renormalizable,i.e. there exists a box B as above, then the width:length ratio of the box is roughly 1:4 and the scalings are as advertized: the scaling in the x-direction is about* $1/16$ *and the scaling in the y-direction close to* $-1/4$.

The map F^2 on B has properties that are identical to F, when instead of the vertical foliation we now look at the foliation $F(\mathbb{H})$ in B. There are a number of ways to straigthen B into a square:

- Via an area preserving transformation. Using this method it is not possible to map the foliation $F(\mathbb{H})$ exactly to $\mathbb{V}$. The resulting renormalized F^2 will then be area preserving. There are many ways to conjugate in an area preserving manner, that preserves the horizontal leaves, via a transformation $(x, y) \to (x + V(y), y)$, composed with a diagonal rescaling. It is possible to choose V so that the fixed point set of RF, $Fix(RF)$ is mapped to the fixed point set of R, $Fix(R)$.
- Preserving the reversible property. Apply the transformation that preserves the horizontal leaves and verticalizes the leafs of $F(\mathbb{H})$. This transformation is essentially canonical.

In both of these cases change coordinates using a diagonal matrix to produce a square as domain for the renomalized maps. When one choose the reversible route, the result is reversible.

Problem 10.5. *Which of these alternatives is preferred?*

Of course there is an alternative method to renormalize, namely as in (Eckmann-Koch-Wittwer formulation, see Gaidashev [44]).

If we choose the **reversible** route, and define that as the renormalization $Reno(F)$ of F:

$$Reno(F) \equiv G^{-1}F^2G$$

By reversibility we have the following property $Reno(F)$ is of the form

$$(x, y) \to (Reno(f)(x, y), x)$$

It is straight forward to deduce an explicit formula for $Reno(f)$.

Furthermore we have:

- Reversibility: $Reno(RF) = R$.
- Invariance: $Reno(FR) = G^{-1}F^2RFG = Reno(F)R$

So in a sense the involution FR behaves like a fixed point under renormalization: at the fixed point of renormalization FR is then fixed. In this formulation something geometric is fixed at as well and that is the axis of symmetry $FRx = x$ of the involution FR. This axis of symmetry is defined for reversible maps, and is a parabola in the Hénon case $x = Q(y)$ with Q quadratic.

We conclude that at the fixed point of renormalization there is special point p_∞ in the Cantor set, and on the axis $FRx = x$ (we note that for every n there are exactly two points in the orbit of p_n on this axis). This pair of points converges to p_∞. This is the point that we will refer as the 'tip' of the Cantor set. It is not exactly the right most point of the Cantor set, but it is canonical.

While $Reno(f)$ maps the vertical foliation to the horizontal, it does not exactly map the horizontal foliation to one that is parabolic. This is not so much a problem, but a fact of life, and it would be interesting to develop a notion of nonlinearity which suitable for this context.

Problem 10.6. *Develop an appropriate notion of nonlinear distortion.*

Even though we have some notion of apriori bounds we believe that it is necessary to move from the reals to two complex variables.

11. Renormalization Operator: Complex

WE suspect that the notion of "Hénon-like" (Romaine DuJardin) is the correct notion case when we consider the Henon map in two complex variables. It is defined on a bi-disk $\mathbb{D} \times \mathbb{D}$ and has the correct algebraic behavior as far as the degree, via a clever definition of 'horizontal', 'vertical' and degree. In particular this notion seems to behave correctly with respect the (relative) count of periodic points contained in the bi-disk. It is difficult to enforce the notion of area preserving in the Hénon-like context. However, the notion of **reversible** may just be good enough.

Theorem 11.1 (Tangerman [38])**.** *There is a natural notion of a holomorphic Renormalization Operator defined on a subset of (reversible) Hénon-like maps, that produces a (reversible) Hénon-like map.*

What is missing in this setting is the notion of a **straightening theorem** as in Douady Hubbard. The situation is however is not entirely hopeless. There are a numbers of notions:

- (Buzzard, Verma: Hyperbolic automorphisms and holomorphic motions in C2) conjugacies can be 'canonically' quasiconformal on leaves of stable or unstable foliations. (usually not on both at the same time in the dissipative case due to additional 'moduli of stability').

  Problem 11.2. *In the reversible case can the conjugacy be chosen so that it is quasiconformal on both laminations? At the same time? Since the qc extension is 'canonical', i.e. Bers-Royden, this could just happen to be the case.*

- Buzzard also has a smooth standard model at infinity that differs from the Hubbard-ObserteVorth model at infinity. It is not exactly holomorphic.

  Problem 11.3. *Might this setting, combined with the notion of Hénon-like allow for an analog of a straightening type theorem? At least in certain cases?*

ADD MORE

12. Aubry Mather Theory

THIS SECTION IS IN NEED OF A PLACE SOMEWHERE. In the real case Aubry-Mather theory is a good tool to prove the existence of invariant sets that are for instance well-ordered, and more.

Problem 12.1. *Is there a variational principle that is appropriate in the holomorphic context, and can explain the existence of KAM annuli, (see below), and ('after bifurcation') the left-over Can-tori for rotation numbers that are sufficiently irrational?*

Problem 12.2. *Same questions, but how about periodic orbits, minima, minimax, etc.?*

Aubry Mather theory also allows one to consider the 'whole' of well-ordered orbits.

Problem 12.3. *What can we say about how periodic well-ordered invariant sets limit on the quasi-periodic ones?*

One can also add stable and unstable manifolds of hyperbolic periodic orbits, see [53, 52, 51].

Problem 12.4. *How, even conceptually, do these limit to the quasi-periodic ones?*

13. The ends of KAM annuli/Herman rings

Given a real, real-analytic, invariant circle C for a holomorphic area preserving map and assume that the rotation number on this circle is sufficiently irrational.

Problem 13.1. *When does there exist a maximal invariant holomorphic annulus A_C, containing C, on which the dynamics is conjugate to a rotation by the given rotation number.*

It is conceivable that there is an invariant set that contains annuli but with ends that are not circles, but rather Cantor sets, like Cantori.

Problem 13.2. *Are there examples where such a largest invariant set is an annulus? For what type of rotation number?*

Conjecture 13.3. *If the rotation number of Diophantine/Bruno then the ends of A_C is a pair of circles, on which the dynamics is transitive* [46].

Problem 13.4. *Are there examples where such a largest invariant set is not an annulus? For what type of rotation number? It is conceivable that the Liouville like rotation numbers that satisfy a Bruno condition, behave entirely unpredictably, and might yield boundaries that are smooth circles. (see Avila-.....)*

Remark 13.5.

- For now we assume that A_C is indeed an annulus.
- Such an invariant annulus A_C is invariant under the operation of complex conjugation
- The intersection with the reals is single Jordan curve C (Why is that?)

Problem 13.6 (Rigidity)**.** *Explain why A_C is an annulus, i.e. has finite modulus, unless the map is integrable.*

Problem 13.7 (Visualize)**.** *How can we visualize such an annulus?*

Problem 13.8. *Is A_C clearly the polynomial (or the 'entire') hull of its two boundary circles?*

Problem 13.9. *Can we think of these annuli as catenoids?*

Problem 13.10. *What can be said about the regularity of the boundary circles?*

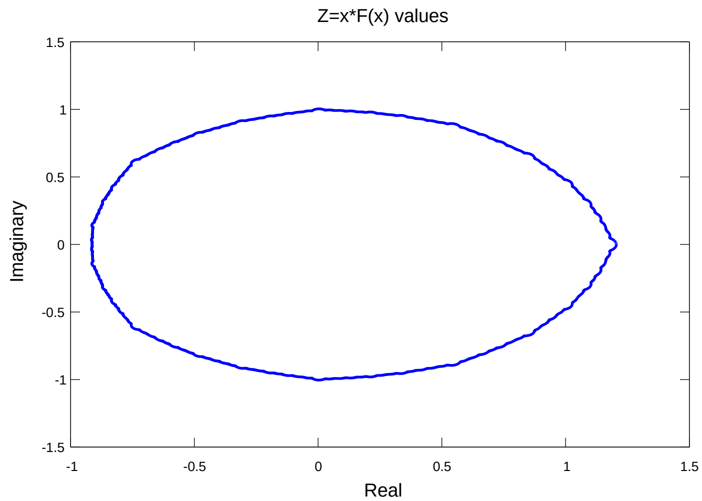

FIGURE 1. *Shown is the image of a circle close to the radius of convergence. It is more or less evident that there are no self-intersections, and that the boundary curve is the image of the boundary of a univalent map. It also shows that the boundary has small detail wiggles on, suggesting that it may not be that smooth.*

Remark 13.11. We are currently investigating an example (the complex standard maps) where instead of annuli one has Siegel disks. For the golden mean rotation number the boundary circle is smooth (see Figure 1), **but not** C^3 and on which the dynamics is **transitive**.

14. Holomorphic Models

In order to have a good understanding how the various sets fit together for complex analytic maps of two variables, it maybe a good starting point, to do this for flows. This is also consistent with results from formal normal forms for mappings which are often time 1 maps of flows.

So a good example to start with is to understand Hamilton Flows in two complex variables:

$$dz/dt = H_w(z,w) \tag{14.1}$$

$$dw/dt = -H_z z, w \tag{14.2}$$

where we take H to be a holomorphic function of two variables, the Hamitonian, and where t may be real or complex time, the last in order to have Riemann surface foliations and laminations.

Problem 14.1. *(also discussed by Camacho Sad) Develop the corresponding theory of Riemann surface foliations.*

Another example is that of a gradient system,

$$dz/dt = F_z(z,w) \tag{14.3}$$

$$dw/dt = F_z z, w \tag{14.4}$$

Where F is a holomorphic of meromorphic function.

Problem 14.2. *Understand the holomorphic Lefschetz formula vor vectorfields in a dynamical fashion.*

Problem 14.3. *Is there analogous definition of Morse function and Morse complex in one or more complex variables?*

Remark 14.4. Recall Milnor's book, *Morse Theory* and Witten's papers: *Supersymmetry and Morse theory* for the real case and the companion paper *Holomorphic Morse Inequalities.* In the latter paper the assumption that the vector field is Killing seems required, but should be inessential. It would be interesting to develop this in the Hamiltonian setting. Also there is possibly an interesting connection with Gromov's (see [50]) theory of pseudoholomorphic curves, genericity and symplectic theory.

Within the context of holomorphic flows as well as in the context formal normal forms one could try to understand the corrsponding foliations.

We next turn to bifurcation theory for flows and mappings.In the real context the corresponding singularity theory has been fully developed by Arnol'd, Thom, Mather, Takens, Zeeman, and many others.

Problem 14.5. *To what extent is the holomorphic bifurcation theory, even for flows or maps in one or two complex variables, already understood?*

Remark 14.6. We note that there was a large Russian school that may have addressed these problems already.

Specifical examples of bifurcations we suggest for study are:

1. Period doubling Bifurcation
2. Saddle node Bifucation
3. Hopf Bifurcation
4. Neimark Sacker bifurcation (used to be called secondary Hopf Bifurcation) in this case for the creation of the Henon dynamics in the complex case.
5. Etc: The field seems to be wide open.

We expect the following objects, Teichmuller theory, and moduli spaces to naturally appear!

Problem 14.7. *Build holomorphic laminations to understand the interaction between KAM theory (and the ends of KAM theory), and hyperbolic theory, as in the real case.*

Problem 14.8. *Introduce Ecalle technology (Borel Summability) to understand parabolic behavior, dynamical parametrization of leaves and the Melnikov method: intersection theory (see Gelfreich and Lazutkin).*

15. Dimensional Reduction and Visualization: Theory of One Complex Variable

Problem 15.1. *How to Visualize these holomophic objects? Did Thom do it? How does Hubbard do it? How do algebraic geometers do it?*

Dimensional reduction can be an important tool for visualization. Assume that a holomorphic system has an invariant curve. If this curve is a partially a graph $w = G(z)$, then there is a corresponding holomorphic dynamics in the z-variable that we can visualize well. Singularities in the projection of the curve become **critical points** of the dynamics in one variable.

Problem 15.2. *Can we apply the theory of one complex variable to not just visualize but also understand the bifurcations (Mandelbrot, Julia sets etc.)?*

Another method for dimensional reduction for flows (and recall that flows often occur for maps are normal forms) is the Poincare section.

Problem 15.3. *Define an appropriate notion of Poincare section in the holormphic context.*

It is sometimes possible to construct a Poincare section, to reduce dimension, and to further reduce.

Problem 15.4. *Explore the following constructions: polynomial hull to construct holomorphic objects, relation with minimal surfaces theory, and the notion of holomorphically connected the refine the notion of connected.*

Remark 15.5. Recall that two points are **holomorphically connected** if there is a chain of holomorphic disks with open non-empty intersection, connecting the two points.

Here is again an interesting problem.

Problem 15.6. *Consider an elliptic island in the reals, it is in a connected component of K. Construct or investigate the lamination(?) with leaves formed by points that are holomorphically connected. Are there transverse measures?*

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