How To Know You're a Geometer

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One evening, I was thinking to myself, "If you add up the entries on the nth row of Pascal's triangle, you get 2^n. I think if you take an alternating sum of the rows, you should get zero."

Just as a reminder, Pascal's triangle begins with the 0th row being "1", the first row being "1, 1", and the second row being "1, 2, 1." The later entries are obtained by adding the two numbers right above. Another viewpoint is that the entries are of the form $\binom{n}{k}$.

Thus, the claim is that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n, \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

The first claim is easy enough to prove: the sum on the left is simply the quantity describing all the subsets of a set of n elements which is 2^n. The second claim is easy to prove as well. However, I didn't immediately see why it's true so I took a different approach.

Consider how the kth Betti number of an n-torus T^n is $\binom{n}{k}$ (one can determine this via Künneth's formula or by choosing the right Morse function on T^n and finding that the chain complex is equal to the homology). It's easy to construct a global, nonvanishing vector field on T^n; simply take the vector field $V = \partial_{x_1}$ on $\mathbb{R}^n$; it descends to a vector field on the quotient $T^n = \mathbb{R}^n/\mathbb{Z}^n$. The obstruction to the existence of such a vector field is the Euler characteristic χ which is precisely given as the alternating sum of Betti numbers. Thus, $\chi = 0$.

I suppose the more elementary way to prove this is to just recall that $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$ as one can see on Pascal's triangle. Then, if we're taking an alternating sum, we'll get something like $1 - (1 + A) + (A + B) - (B + C) + (C + D) - ... = 0$.