The Ants and the Grasshopper Live on a 3-Manifold

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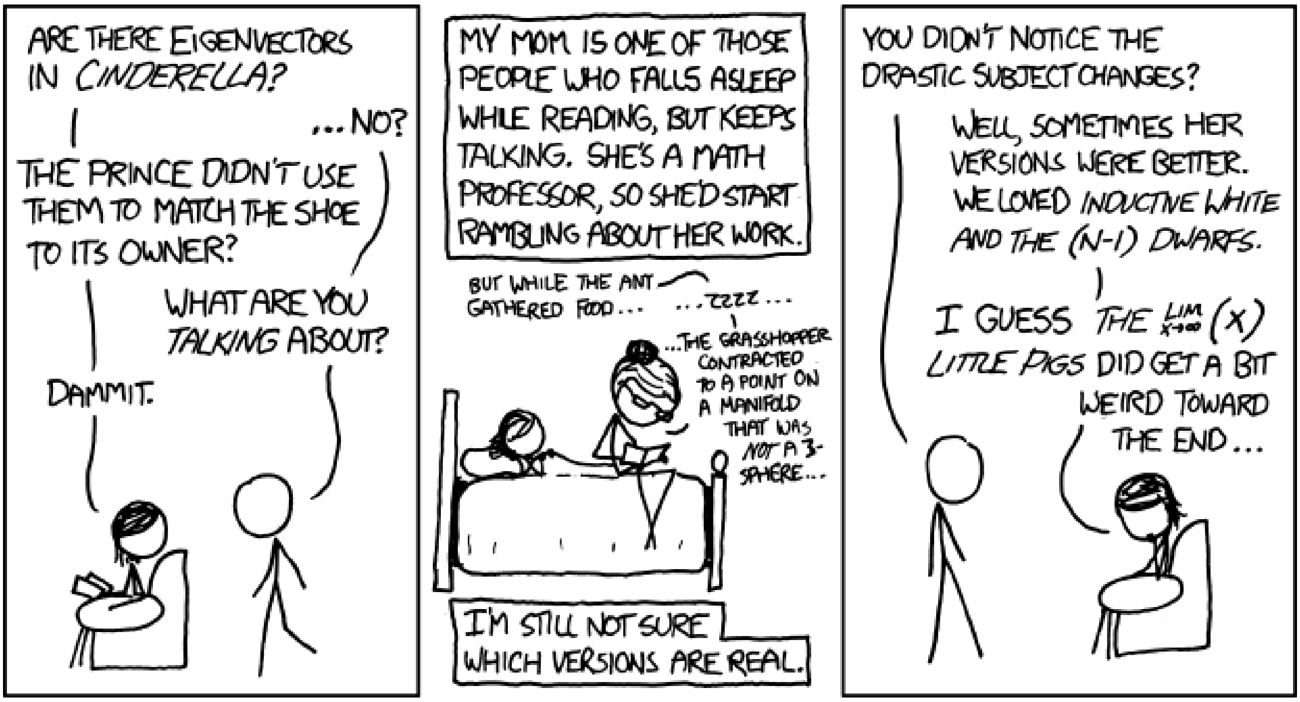
Randall Munroe, the creator of the popular webcomic xkcd has this comic about fairy tales and math mixing together. This is quite an old webcomic, number 872. However, when I looked to see if anyone has actually written a version of the fairy tale, I didn't find any. Of course, I didn't look that hard; I was already thinking of my own version of the story.



Thus, I decided to try writing a few of my own spins on classic fairy tales. I tried to stick somewhat close to the original stories. This is surely contestable because in the case of Cinderella, say, there are many versions, including the Brothers Grimm and also Disney. I would lean towards the Disney version both because it's more well-known and also, less gruesome. I also wanted to stick to Munroe's original idea. I'm not sure how to incorporate eigenvectors into Cinderella but there you go.

I had thought the fable of the ant and the grasshopper may be one of Aesop's but like any tale, there are many versions. The fable is quite short. We have a grasshopper who spends the summer singing and, in the version I heard growing up, mocks the working ants who store up food for the winter. When winter comes and lasts longer than the Game of Thrones episode in which winter finally arrives (I've been told), the grasshopper shivers in the cold and begs the ants for food. However, the ant rebukes the grasshopper for idleness and tells it to dance the winter away. I suppose the grasshopper perishes or, in happier versions, survives the winter and learns the lesson.

This one is a bit hard. Munroe has the mother saying that the grasshopper contracts to a point on some 3-manifold which is not the 3-sphere. I don't know what contracting to a point should mean in the context of the fable. I take it to literally meaning there is a homotopy equivalence between the grasshopper and a point. Also, whatever manifold they're living on, it can't be simply connected, by the Poincaré conjecture (now theorem). Off the top of my head, I can't name that many closed, orientable 3-manifolds. S^3, $S^1 \times \Sigma$ where Σ is a Riemann surface, homology 3-spheres, and $\mathbb{R}P^3$. I opted for $M = S^1 \times S^2$.

1 The Fable

Once upon a time, there was a grasshopper who loved to frolic in the summer breeze and sing. As you may know, grasshoppers have their ears on their abdomens and can vibrate their wings very quickly to make noise. Possibly, that's the music this grasshopper sang.

One day, the grasshopper chanced upon a colony of ants. Sidling up to one of the ants, he said, "Hey there, friend. What's your name? And why don't you join me in a song?" The ant did not stop using its mandibles to drag the grain along. "The name's Perelant. We're storing up food for the coming winter."

The grasshopper scoffed, "What is this winter? What it could it possibly do that would require such hard work?" Perelant replied, "Why, it can become very cold. So cold in fact, that the world undergoes a huge transformation. There will be snow, subzero temperatures (ants use Celsius, of course, even ones in the US), and—"

"Pshhaww!" the grasshopper, interrupted. "I know about that. It's nothing to be afraid of. For I can hide from the snow." At this, the grasshopper contracted to a point. "See?" said the point. "I can hide really well. I find a noncontractible loop to hide in and the snow can't stick to it; snow needs something at least two dimensional."

The ant shook its head and went about its work. Perelant knew, as did all ants, because they wander far, that their world was $S^1 \times S^2$. The grasshopper was hiding on the S^1 component and hence, has noncontractible loops. But the grasshopper, always idle, had not explored the global structure of the manifold.

Time passed and summer turned to autumn. Still, the grasshopper sang its song and every now and then, contracted to a point to show off, while the ants worked. Even when the gray of winter appeared, the grasshopper was unperturbed. "I'll just contract to a point!" the grasshopper said to itself.

However, this winter was especially cold. And when it hit zero degrees, something extraordinary happened. The world was hit by a zero surgery performed along an unknot. Suddenly, the world was now S^3!

The grasshopper, in the subzero temperatures, tried contracting to a point. But there was no longer any noncontractible 1-cells to hide under. "Where has the topology gone?" cried the grasshopper. "Why, there are no 1-cycles nor 2-cycles left!" The poor grasshopper had no choice but to call on the ant colony.

"Perelant! Please, let me into your home. Please give me some grain to eat."

Unlike Aesop's fable, the ant at the door, Hamiltant, let the grasshopper in but the welcome was accompanied by some stern lecture on idleness. "You must apply your mind and body," said Hamiltant. The grasshopper nodded but had a more pressing question on its mind. "How is your home warm, anyways? I traveled around and discovered that this world has no boundary! Not like that weird world we hear about in stories which is basically B^3. What's protecting us from the snow?"

The wise ants told the grasshopper the secret. "When an especially hard winter comes, it changes our world from $S^1 \times S^2$ to S^3, via zero surgery. However, the space is split into three parts: two pieces are solid tori $S^1 \times D^2$ and $D^2 \times S^1$; they have a common boundary T^2. This

T^2 is what determines what is "underground" and what is above ground. We are now in the underground solid torus where its warmer."

"Wow " exclaimed the grasshopper. "How did you figure that out?" Perelant replied, "Well, there once was an ant named Heegaard who found that a natural self-indexing Morse function f on $S^1 \times S^2$ gives us a regular hypersurface $f^{-1}(3/2)$. This is precisely the torus I mentioned. When zero surgery comes, well, this torus is preserved but..." And Perelant went on explaining as the grasshopper, whose name is Thurston, munched on grains. The end.

2 Comments

I had initially wanted to try writing a story involving Ricci flow since the coming of winter deals with heat loss and Ricci flow is formally like heat flow. But I couldn't think of a story. So I opted to name some ants after Richard Hamilton and Grigori Perelman. I thought it would be fun to name the grasshopper after William Thurston though in the story of 3-manifolds, Hamilton and Perelman initially learned from Thurston's ideas rather than the other way around. The ants let the grasshopper in because, in general, mathematicians are quite open to collaborating and talking about ideas.

The weird world that Thurston mentions, B^3, is really our world. William Thurston, from what I hear, does not have the personality of this grasshopper. Similar for Perelman and Hamilton.