AN INTRODUCTION TO WEINSTEIN MANIFOLDS

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Weinstein manifold is a class of open, exact symplectic manifolds. Important examples include cotangent bundles of closed smooth manifolds, as well as Stein manifolds (e.g. smooth affine varieties). In fact, under some suitable generalizations, *both* of these examples are universal (in some technical sense).

This note heavily borrows from [5] and Chapter 11 of [3].

1. From Liouville to Weinstein

By Stokes' theorem, a symplectic manifold (V, ω) with an exact symplectic form has to be open. We first assume that V is compact. Then V has non-empty boundary. Taking a primitive λ of the symplectic form ω (i.e. $d\lambda = \omega$), the vector field X that is ω-dual to λ (i.e. $\iota_X \omega = \lambda$) is called the **Liouville vector field**.

Definition 1.1. *A* ***Liouville domain*** *is a compact exact symplectic manifold (V, ω) with a choice of primitive λ such that one of the equivalent conditions hold:*

- *$\lambda|_{\partial V}$ is a contact form on ∂V, and the orientation of ∂V given by the contact form coincides with the orientation as the boundary of the V (whose orientation is given by the symplectic structure).*
- *The Liouville vector field X is transverse to ∂V, and points outwards.*

The condition that $\iota_X \omega = \lambda$ implies $\mathcal{L}_X \lambda = \lambda$, so the positive flow expands ω: if ϕ_t is the flow of X, then $\phi_t^* \lambda = e^t \lambda$, and thus $\phi_t^* \omega = e^t \omega$.

The condition that X is transverse to ∂V is good for pseudo-holomorphic curve purposes, but it is also useful for a more elementary reason: it gives us a "symplectic tubular neighborhood" of the boundary. Then, given a Liouville domain (V, ω, X), there is a standard way of "completing" it: the condition that X is outward transverse to the boundary implies that there is a symplectic collar neighborhood $\partial V \times (-\epsilon, 0]$ with $\lambda|_{\partial V \times \{t\}} = e^t \lambda|_{\partial V \times \{0\}}$. Therefore we can attach a cylindrical end $\partial X \times [0, \infty)$ with Liouville 1-form given by the same formula $\lambda|_{\partial V \times \{t\}} = e^t \lambda|_{\partial V \times \{0\}}$ to get a non-compact manifold $\hat{V} = V \cup (\partial V \times [0, \infty))$ with complete Liouville vector field (notice that the cylindrical end we attached is half of the symplectization of ∂V). This is a **Liouville manifold**, i.e. an exact symplectic manifold with a complete Liouville vector field and an exhaustion by (compact) Liouville domains. We often think about Liouville manifolds and Liouville domains interchangeably: **finite-type** Liouville manifolds, which are Liouville manifolds with compact skeleton (defined below), corresponds to completion of Liouville domains.

An important notion for a Liouville domain (V, ω, X) is the **skeleton** of the Liouville flow:

$$\mathrm{skel}(V, \omega, \lambda) = \bigcap_{t > 0} \phi_{-t}(V).$$

This is the attractor of the negative flow of X. A point $x \in V$ is in the skeleton if and only if it does not flow "out" of the domain. Given a Liouville manifold $\tilde{V}$ with exhaustion $\tilde{V} = \bigcup V_i$ by Liouville domains V_i, we define its skeleton to be the union of $\mathrm{skel}(V_i)$. Notice that since the negative flow of the Liouville vector field X exponentially contracts the symplectic form, and thus the volume form, the skeleton has volume 0. However, the skeleton might still be "big", as McDuff [9] has constructed Liouville domains such that

- $\mathrm{skel}(V)$ has codimension 1;
- ∂V has more than one components.

On the other hand, a Weinstein manifold, which is a Liouville manifold whose Liouville flow is tamed by a Morse (or generalized Morse) function, does not have these behaviors (Weinstein domains could have disconnected boundary in dimension 2, but this is impossible higher dimensions).

Definition 1.2. *A* ***Weinstein domain*** *is a Liouville domain (V, ω, X) together with a Morse function $\phi : V \to \mathbb{R}$ satisfying*

$$d\phi(X) \geq \delta(\|X\|^2 + \|d\phi\|^2) \tag{1}$$

for some $\delta > 0$. A pair (X, ϕ) satisfying (1) is said to be a Lyapunov pair; ϕ is called a Lyapunov function for X, and X is called gradient-like for ϕ.

Similarly, a ***Weinstein manifold*** *is a Liouville manifold (V, ω, X) together with a Morse function $\phi : V \to \mathbb{R}$ that is*

- *exhausting, i.e. proper and bounded from below;*
- *Lyapunov for X, i.e. there exists $\delta : V \to \mathbb{R}_+$ such that*

$$d\phi(X) \geq \delta(\|X\|^2 + \|d\phi\|^2).$$

Remark 1.1.
(1) The Lyapunov condition (1) implies that the zeroes of X exactly correspond with the critical points of ϕ. One reason tha Weinstein manifolds are better behaved than Liouville manifolds is that the local theory of zeros of vector fields become much simpler if there is a taming function.
(2) The taming function ϕ does not need to be Morse for most purposes – generalized Morse (allowing birth-death type critical points) is usually enough. But for simplicity we restrict attention to Morse functions.
(3) A Weinstein manifold is of **finite-type** if ϕ has finitely many critical points. Finite-type Weinstein manifolds correspond to completions of Weinstein domains.
(4) In some place (1) is written as

$$d\phi(X) \geq \delta\|X\|^2.$$

This is equivalent to (1) if the critical points of ϕ are non-degenerate. See Remark 9.11 in [3].

Example.
(1) $\mathbb{C}^n$, with standard symplectic structure $\omega_{std} = \sum dx_j \wedge dy_j$, with Liouville vector field and Weinstein Morse function

$$X = \frac{1}{2}\sum_j \left(x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j} \right), \quad \phi = \frac{1}{4}\sum_j \left(x_j^2 + y_j^2 \right) = \|z\|^2.$$

(2) T^*Q for Q a closed smooth manifold. The "obvious" choice is to choose λ_{std} the tautological 1-form (locally, $\lambda_{std} = p\,dq$), which gives Liouville vector field $X = p\frac{\partial}{\partial p}$, and the taming function $\phi = \|p\|^2/2$. There are two problems: the taming function is Morse-Bott instead of Morse (since the entire zero section are critical points), and more importantly we hope our function to reflect the topology of Q.

The "better" way to do this (see [3], Example 11.12 (2)) is to pick a Morse function $f : Q \to \mathbb{R}$, and let

$$\phi(q,p) = \frac{1}{2}\|p\|^2 + f(q),$$

Then the critical points of ϕ on T^*Q are exactly those of f on Q. Now let $F = \lambda(\nabla f)$ where ∇f is the gradient of f (with respect to some fixed Riemannian metric), and let X_F be the Hamiltonian vector field of F. Then X_F is close to ∇f near Q and $X_F = \nabla f$ on Q. We take the Liouville vector field to be

$$X = p\frac{\partial}{\partial p} + X_F.$$

Notice that the skeleton of a cotangent bundle T^*Q is the zero section Q.

(3) Stein manifolds, including smooth affine varieties. We defer this to section 3.

Before we move on to the next section, we mention a quite surprising result, which is proved using quite elementary methods:

Proposition 1.1 (Lemma 11.2 of [3])**.** *Any symplectomorphism between finite-type Liouville manifolds is isotopic (through symplectomorphisms) to an exact symplectomorphism.*

Recall the nearby Lagrangian conjecture:

Conjecture 1.1 (Nearby Lagrangian Conjecture)**.** *If Q is a closed smooth manifold and L a closed exact Lagrangian in T^*Q, then L is Hamiltonian isotopic to the zero section (and is, in particular, diffeomorphic to Q.*

We want to remark that, assuming nearby Lagrangian conjecture is true, the previous proposition implies that the symplectic geometry of T^*Q uniquely determines the smooth topology of Q: if $\varphi : T^*Q_1 \to T^*Q_2$ is a symplectomorphism, since T^*Q_i have skeleta Q_i which are compact (and therefore T^*Q_i are of finite type), φ can be upgraded to an exact symplectomorphism. Therefore the image of the zero section of T^*Q_1 is a closed exact Lagrangian in T^*Q_2, so it is Hamiltonian isotopic to the zero section Q_2. Therefore Q_1 and Q_2 are diffeomorphic.

2. From Morse/Smale to Weinstein

In smooth topology, given a smooth manifold Q and a Morse function $f : Q \to \mathbb{R}$, we can do Morse theory on Q, e.g. defining the Morse homology, with the gradient ∇f of f, if the pair $(f, \nabla f)$ satisfies certain transversality properties (which is generic). But we don't have to stick with the gradient vector field – it suffices for the vector field to be "gradient-like" (see e.g. [2, 10]). For example, we can modify the vector field outside some small neighborhood of critical points, as long as it is going towards the "correct direction". We call such pair (ϕ, X) a Morse-Smale pair if the necessary transversality properties are satisfied.

Along this line, we can think of a Weinstein manifold as a manifold with a Morse-Smale pair (ϕ, X), with X compatible with the symplectic structures. Indeed, the name Weinstein manifold comes from Weinstein's paper [12] studying symplectic handlebodies. This pair will allow us to do a lot of the usual smooth Morse theory and handle calculus symplectically.

The point of Morse-Smale theory in smooth topology is to break a manifold into simple pieces. The way we do this is to take a Morse function $\phi : M \to \mathbb{R}$, modify so that ϕ has a different value on each critical point, and take the cobordisms $\phi^{-1}[a_i, b_i]$ where each $\phi^{-1}[a_i, b_i]$ contains exactly one or no critical points, and each of these "elementary cobordisms" are standard pieces that we understand well. We want to do the same thing for Weinstein manifolds; but we must carry the geometric structures around and worry about how they glue together. Thus we first need to understand the symplectic structures on these elementary pieces.

We work with **Weinstein cobordisms** (W, ω, X, ϕ), which has two components of the boundary $\partial_+ W$ and $\partial_- W$, with the Liouville vector field pointing outward (resp. inward) at $\partial_+ W$ (resp. $\partial_- W$). We also require that $\partial_\pm W$ are level sets, say $\partial_- W = \phi^{-1}(a)$ and $\partial_+ W = \phi^{-1}(b)$.

Suppose ϕ has no critical points on W. Then the Liouville field X also has no zeroes. Complete W at the $\partial_+ W$ end to get $\hat{W}$. Define $\phi^{-1}(a) \times [0, \infty) \to \hat{W}$ by flowing along X. This map embeds the symplectization of $\phi^{-1}(a)$ in $\hat{W}$. Then $\partial_+ W = \phi^{-1}(b)$ is the graph of some function $f : \phi^{-1}(a) \to \mathbb{R}$ inside the cylinder $\phi^{-1}(a) \times [0, \infty)$. Let $\Phi : \phi^{-1}(a) \to \phi^{-1}(b)$ be defined by the flow. Then $\Phi^* \lambda_{\phi^{-1}(b)} = e^f \lambda_{\phi^{-1}(a)}$. So Φ is a contactomorphism. In particular,

Lemma 2.1 (Lemma 11.13 (b) in [3])**.** *Let (V, ω, X, ϕ) be a Weinstein manifold. Suppose ϕ has no critical values in $[a, b]$. Then the image of any isotropic submanifold $\Lambda^a \subset \phi^{-1}(a)$ under the flow of X intersects $\phi^{-1}(b)$ in an isotropic submanifold $\Lambda^b \subset \phi^{-1}(b)$.*

Suppose ϕ has exactly one critical point. Then:

Proposition 2.1 (Lemma 11.13 (a) in [3])**.** *Let (V, ω, X, ϕ) be a Weinstein manifold. The stable/descending manifold W_p^- of any critical point $p \in V$ of ϕ satisfies $\lambda|_{W_p^-} \equiv 0$. In particular, W_p^- is an ω-isotropic submanifold of V, and $W_p^- \cap \phi^{-1}(c)$ is a λ-isotropic submanifold of the contact manifold $\phi^{-1}(c)$ for any regular value c.*

Proof. Take any $q \in W_p^-$, and $T_v W_p^-$. Let ϕ_t be the time-t flow of the Liouville field. We know that $\phi_t^* \lambda = e^t \lambda$, so

$$e^t \lambda_q(v) = \phi_t^* \lambda_q(v) = \lambda_{\phi_t(q)}[(\phi_t)_* v].$$

Since $\phi_t(q) \to p$ as $t \to \infty$, so

$$\lambda_q(v) = e^{-t} \lambda_{\phi_t(q)}[(\phi_t)_* v] = 0.$$

□

By Proposition 11.9 (c) of [3], the unstable manifold W_p^+ is coisotropic. In handle calculus language,

- W_p^+ is the (coisotropic) cocore;
- W_p^- is the (isotropic) core;
- $W_p^- \cap \phi^{-1}(c)$ is the (isotropic) attaching sphere.

Remark 2.1. (1) Since $\text{Zero}(X) = \text{Crit}(\phi)$, the skeleton of (V, ω, X) is exactly the union of all stable manifolds. By the proposition, the stable manifolds are all isotropic, so the skeleton must have dimension at most n (if the Weinstein manifold has dimension $2n$).

(2) The skeleton of a Liouville domain is a deformation retract of the domain. We have shown that the skeleton is isotropic for Weinstein domains, so there is no symplectic topology happening in the skeleton. It is expected that a lot of the symplectic topology of the Weinstein domain is contained in the "smooth topology" (of course, the skeleton can be singular) of the skeleton.

In Morse theory, the index of a critical point equals the dimension of corresponding stable manifold. Since isotropic submanifolds have dimension at most n (say, for a finite-type Weinstein manifold of dimension $2n$), we have :

Corollary 2.1. *A finite-type Weinstein manifold of dimension $2n$ has the homotopy type of a finite CW complex of dimension at most n.*

Assuming we know that all Stein manifolds are Weinstein (we will define a Stein manifold and show that they are Weinstein in section 3, or see [3] section 11.5), we have given a symplectic proof of:

Corollary 2.2. *A finite-type Stein manifold, and therefore a smooth affine variety (see [11] 4b), has the homotopy type of a finite CW complex of dimension at most n.*

In [10], the proof of h-cobordism theorem depends on several geometric results: a rearrangement theorem that allows us to reorder critical points, an isotopy extension result that allows us to isotope attaching spheres, and theorems on criterion for creation and cancellation of a pair of critical points. These results turn out to hold in Weinstein handle calculus. These four results are listed in 12.6 of [3]. We give the ideas of these results in handle language (these are called Weinstein homotopy moves):

(1) An isotropic isotopy of the attaching sphere does not affect the result of handle attachment;
(2) If the belt sphere and attaching sphere of two handles are disjoint (i.e. the differential between these two critical points is zero in Morse homology), then they can be reordered;
(3) If the attaching sphere of a $(k+1)$-handle and the belt sphere of a k-handle intersect transversely exactly once (i.e. the $(k+1)$ critical point has exactly one negative gradient flow line that reaches the k critical point, so that they cancel in Morse homology), then these two handles can be "cancelled". Conversely, we can "create" two handles of index $k+1$ and k with this property.

Now we take a more constructive viewpoint. Given a Weinstein domain and a isotropic sphere in the boundary, when can we attach a symplectic handle?

In smooth topology, the handle calculus implicitly depends on the tubular neighborhood theorem, which tells us near each sphere in the boundary its neighborhood looks "essentially the same", so after choosing a framing the procedure of doing the handle attachment inside this neighborhood is standard. So in order for symplectic handle calculus to work, the first step is to prove an isotropic neighborhood theorem. Given a contact manifold $(M, \xi = \ker \alpha)$ and an isotropic submanifold L in M, we define the **conformal symplectic normal bundle**

$$CSN_M(L) = (TL)^{\perp}/TL$$

where $(TL)^{\perp}$ is the symplectic orthogonal complement of TL in the contact distribution ξ. Notice that the topological normal bundle NL can be decomposed into

$$NL \cong \langle R \rangle \oplus J(TL) \oplus CSN_M(L) \quad (2)$$

where $\langle R \rangle$ is the trivial line bundle spanned by the Reeb vector field, and J an $d\alpha$-compatible almost complex structure on ξ.

Theorem 2.1 (Contact isotropic neighborhood theorem, see [6] Corollary 6.2.2)**.** *Let (M_i, α_i), $i = 0, 1$ be strict contact manifolds with closed isotropic submanifolds L_i. Suppose there is an isomorphism of symplectic normal bundles $\Phi : CSN_{M_0}(L_0) \to CSN_{M_1}(L_1)$ that covers a diffeomorphism $\Phi : L_0 \to L_1$. Then this diffeomorphism Φ extends to a strict contactomorphism $\psi : \mathcal{N}(L_0) \to \mathcal{N}(L_1)$ of suitable neighbourhoods $\mathcal{N}(L_i)$ of L_i such that $T\psi|_{CSN_{M_0}(L_0)} = \Phi$.*

There is one more issue related to the **framing** of a handle. The attaching *region* of a k-handle is a copy of $S^{k-1} \times D^{2n-k}$, so in order to specify the gluing of the handle, we need to specify an embedding not only of an isotropic S^{k-1}, but an embedding of $S^{k-1} \times D^{2n-k}$. Equivalently, we choose a trivialization of the normal bundle of the isotropic sphere. Now we claim that in the splitting (2), the $\langle R \rangle \oplus J(TL)$ vector bundle is *trivial* if L is an isotropic sphere. Now notice that, as an embedded sphere $S^{k-1} \subset \mathbb{R}^k$, the direct sum of TS^{k-1} with a trivial line bundle has a natural trivialization. So roughly speaking, an identification of L with $S^{k-1} \subset \mathbb{R}^n$ gives a trivialization of $\langle R \rangle \oplus J(TL)$. This is why specifying a trivialization of $CSN_M(S^{k-1})$ is sufficient for choosing a framing; the framing chosen in this way is called the **natural framing** determined by the choice of trivialization of $CSN_M(S^{k-1})$.

One important consequence of the above discussion is that for a Legendrian sphere in a $2n-1$-dimensional contact manifold, $CSN_M(S^{n-1})$ has rank 0. Therefore one can always attach a symplectic handle along a Legendrian sphere, and specifying an embedding of a Legendrian sphere suffices to completely determine the handle attachment. So the subcritical handle attachment is "inessential" for a lot of symplectic purposes, and critical handle attachment is cleanest with respect to framing issues.

Now we can state the contact surgery/symplectic handle attachment theorem, first proved by Weinstein in [12]. We will leave the standard details of e.g. smoothing corners to e.g. [6].

Theorem 2.2. *Let S^{k-1} be an isotropic sphere in a contact manifold $(M, \xi \ker \alpha)$ with a trivialisation of the conformal symplectic normal bundle $CSN_M(S^{k-1})$. Then there is a symplectic cobordism from (M, ξ) to the manifold M' obtained from M by surgery along S^{k-1} using the natural framing. In particular, the surgered manifold M' carries a contact structure that coincides with the one on M away from the surgery region.*

Thus, we can essentially define Weinstein domains as symplectic handlebodies.

Example. We give a handle decomposition of the T^*S^n. We claim that this is given by attaching an n-handle on the standard $2n$-ball D^{2n} (with coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$), along the sphere

$$S^{n-1} = \left\{ (x_1, \ldots, x_n, 0, \ldots, 0) \mid \sum_{i=1}^n x_i^2 = 1 \right\}$$

in the boundary S^{2n-1} of D^{2n}. For $n = 2$ this attaching sphere is the Legendrian unknot in S^3.

The key point is the following:

- The attaching Legendrian sphere bounds a Lagrangian disk

$$D^n = \left\{ (x_1, \ldots, x_n, 0, \ldots, 0) \mid \sum_{i=1}^n x_i^2 \leq 1 \right\}.$$

- The core of the n-handle is another Lagrangian disk D^n

These two Lagrangian disks piece together into a *Lagrangian* S^n inside the $D^{2n} \cup_{S^{n-1}} (n\text{-handle})$. By Weinstein neighborhood theorem, a neighborhood of this S^n looks like T^*S^n. Moreover the Liouville fields outside this neighborhood have no zeroes, so attaching a n-handle along the unknotted S^{n-1} in the boundary of B^{2n} exactly gives us T^*S^n.

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