Hamiltonian Floer theory:

Let (M^{2n},ω) - closed symp mfld & $H: M \to \mathbb{R}$ - Hamiltonian. Since ω-nondeg, $\exists!$ ω-dual to dH, a vect field X_H.

Locally, M is $(\mathbb{R}^{2n},\omega_0)$ - position-momenta space & X_H gives dynamics/laws of motion determined by

$\dot{\gamma} = X_H(\gamma)$; Note: $\underset{0}{\underset{\shortparallel}{\omega(X_H,X_H)}} = dH(X_H) \Rightarrow X_H$ preserves level sets of H; view as preserving energy; conservation of energy.

i.e. $H \circ \varphi_t^H = H$

Arnold: Are there orbits of X_H? If so, how many?

Conjecture: For nondegenerate H_t (time dependent, which is most H), the # of 1-periodic orbits $\geq \sum b_i(M)$. So this conjecture predicts more than classical topology's prediction of $\sum (-1)^i b_i(M) = \chi(M)$. (Lefschetz fixed pt thm)

We can view 1-periodic orbits as fixed pts of φ_1 - time 1-flow.

Rough Idea: Let $\mathcal{L}M$ = loop space of contractible loops. Do "Morse theory" on $\mathcal{A}_H(\gamma) = -\int_{D^2} \bar{\gamma}^*\omega - \int_{S^1} H_t(\gamma(t))\,dt$ b/c (need some conditions like aspherical)

$(d\mathcal{A}_H)_\gamma(\xi) = -\int_{S^1} \omega(\xi, \dot{\gamma}(t) - X_H(\gamma))\,dt$

$= 0 \iff \dot{\gamma} = X_H(\gamma)$.

So Ham 1-per orbits are exactly the crit pts.

Let $CF_*^*(M,H)$ be generated by over $\mathbb{Z}/2$.

∂ defined by count of $u: \mathbb{R}_s \times S^1_t \to M$ satisfying $\partial_s u + J\partial_t u = \nabla H$

$\omega(\cdot, J\cdot)$ gives metric.

J-generic

Need Conley-Zehnder index.

Pf of Arnold's Conjecture: $HF_*^*(M,H) \overset{\text{continuation maps}}{\cong} HM_*^*(M,f) \cong H_*(M)$. ∎

Arnold's conjecture: $\exists$ orbits of given period

Weinstein conjecture: $\exists$ orbits of given energy level (when the energy level is contact type).

Defn: $\Sigma \subset (M,\omega)$ is a contact hypersurface if $\exists$ vect field X $\pitchfork \Sigma$ (defined in nbhd of Σ) s.t. $\mathcal{L}_X\omega = \omega$. $\lambda = \iota_X\omega$ is called the Liouville 1-form.

Defn: If (M,ω) has ∂M = contact type, we want X to be outward pointing. Suppose also that X is globally def. Then $\omega = d\lambda$ & we call (M,ω) a Liouville domain.

We have R - Reeb field on $\overset{\Sigma}{\partial M}$ & completion $\hat{M}$ (Shuhao should discuss these the week prior).

A nbhd of $\Sigma = \partial M$ is foliated & $\cong (\Sigma \times [1-\delta, 1+\delta]_t, d(e^t\lambda))$; $X_H \in \ker \omega|_{T\Sigma}$ (reg level set) & $\lambda(X_H) \neq 0$.

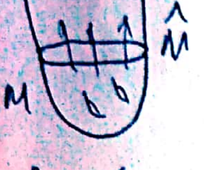
If $H: M \to R$ has Σ as level set, the orbits of X_H on Σ $\longleftrightarrow$ orbits of R. Hence, dynamics of H don't really depend on H but on Σ!

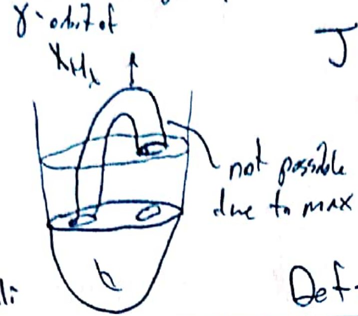
So we would like to choose "admissible" $H_\lambda : \Sigma \times [1-\delta, \infty)_t \longrightarrow \mathbb{R}$ w/ $H(p,t) = h_\lambda(t)$, $h_\lambda : [1-\delta, \infty) \to \mathbb{R}$ where $h'' > 0$ & $h_\lambda = \lambda t$ on $[1, \infty)$. Then $X_{H_\lambda}(p,t) = -h'(t) R$, so {1-periodic orbits of X_H on level t} $\longleftrightarrow$ {Reeb orbits of period $-h'(t)$ on Σ}.

For large λ, we see lots of orbits. Extend H as a small Morse f^n to M.

Def: $SC^*(M, H_\lambda, \mathbb{K}) = \bigoplus \mathbb{K}\langle\gamma\rangle$, coeff ; let J be a cylindrical ACS; i.e. compatible w/ ω & $JX = R$. ; differential ∂ counts solutions $u : \mathbb{R}_s \times S^1_t \longrightarrow \hat{M}$ to $\partial_s u + J(\partial_t u - X_{H_\lambda}) = 0$. where $\lim_{s \to \pm\infty} u(s,t) = \gamma^\pm(t)$

Liouville vect field





Our choices allow $\partial^2 = 0$; we have compactness of moduli

Def: $SH^*(M, \mathbb{K}) \doteq \lim_{\lambda \to \infty} SH^*(M, H_\lambda, \mathbb{K})$.

Properties:

0. $SH^*(M, \mathbb{K})$ is $\mathbb{Z}/2$-graded; if $c_1(M) = 0$, we have $\mathbb{Z}$-grading (requires choices unless $H^1(M) = 0$)

1. $\exists\ H^{*+n}(M) \longrightarrow SH^*(M)$; its failure to be an isomorphism $\Rightarrow \exists$ a periodic Reeb orbit in $\Sigma = \partial M$

2. If the Reeb flow gives a free circle action on ∂M, $c_1(M) = 0$, & ∂M is connected, $\exists$ Morse-Bott Spectral seq

$$E_1^{p,q} = \begin{cases} H^{q+n}(M, \mathbb{K}) & , \quad p = 0 \\ H^{p+q+n-p\mu}(\partial M, \mathbb{K}) & , \quad p < 0 \\ 0 & ; \quad p > 0 \end{cases} \Rightarrow SH^*(M).$$

For $M = B^{2n}$, $\mu = 2n$, the differential d_1 is acyclic so $SH^*(B^{2n}) = 0$

3. $SH^*(M \times N) \cong SH^*(M) \otimes SH^*(N)$

4. (Cieliebak): Subcrit Weinstein domains M is def equiv to $N \times \mathbb{C}$, hence by Künneth, $SH^*(M) = 0$.

5. If ω_s is an isotopy of Liouville forms on M, then $SH^*(M, \omega_0) \triangleq SH^*(M, \omega_1)$.
In particular SH^* is independent of the contact form on ∂M, just depends on the contact struc. (mean euler char is a contact invariant of ∂M)

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e.g. $SH^*(\mathbb{C}^*) \cong \mathbb{Z}[x, x^{-1}]$, $|x| = 1$. So SH^* is not bounded from above or below in degree.

Caveat: What Seidel calls cohomology is defined by a direct limit & the differential has deg -1, so it appears more like homology.

Viterbo Functoriality

Suppose $W \xhookrightarrow{j} M$ is a codim 0 embedding of a Liouville subdomain. Also, suppose $\exists$ ACS J s.t. all its Floer traj limiting to orbits in W, stay in W. Then we have the following diagrams

$$\begin{tikzcd} SH^*(M) \arrow[r, "F_j^*"] & SH^*(W) \\ H^{n+*}(M) \arrow[u, "c^*"] \arrow[r, "j^*"] & H^{n+*}(W) \arrow[u, "c^*"] \\ H_{n-*}(M,\partial M) \arrow[u, phantom, "\cong" rotate=90] & H_{n-*}(W,\partial W) \arrow[u, phantom, "\cong" rotate=90] \end{tikzcd}$$

; F_j^* is a unital ring morph

dually $\rightsquigarrow$

$$\begin{tikzcd} SH_*(M) \arrow[d, "c_*"] & SH_*(W) \arrow[l, "F_j!"'] \arrow[d, "c_*"] \\ H_{n-*}(M) & H_{n-*}(W) \arrow[l, "j!"'] \end{tikzcd}$$

where F_j^*, $F_j!$ are defined by certain curve counts.

↗ View as low energy orbits (Morse theory)

So $c_*: SH_*(M) \longrightarrow SH_*(M)_{(a,\varepsilon]}$ $a<0$, $\varepsilon>0$ small
$\cong$
$H^{*+n}(M,\partial M)$

is induced from restriction

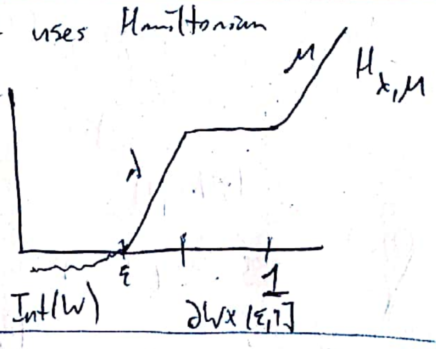
Thom iso $S \xhookrightarrow[\text{codim } k]{j} M$, $M \xrightarrow{\text{collapse}} M/(M\setminus \underset{\substack{\uparrow \\ \text{tubular} \\ \text{nghd}}}{N(S)}) = Th(N(S))$

gives

$$\begin{tikzcd} H^{*-k}(S) \cong H^*(Th(N(S))) \arrow[r] & H^*(M) \end{tikzcd}$$

with $H^{*-k}(S) \xrightarrow{j!} H^*(M)$

$F_j!$ uses Hamiltonian



Cor: Suppose $SH^*_n(M) \xrightarrow{c_*} H^{2n}(M,\partial M) \cong \mathbb{Z}/2$ is not surj. Then $\nexists L \hookrightarrow M$, exact Lagrangian.

pf: Suppose $\exists$ such $L \subset M$. By Weinstein nghd thm, $D^*L \xhookrightarrow{j} M$ so we have the diagram.

$$\begin{tikzcd} SH_n(D^*L) \arrow[r] \arrow[d, "c_*"] & SH_n(M) \arrow[d, "c_*"] \\ H^{2n}(D^*L,\partial D^*L) \arrow[r, "j!"] & H^{2n}(M,\partial M) \end{tikzcd}$$

Note: $H^{2n}(D^*L,\partial D^*L) \underset{\text{Thom}}{\cong} H^n(L)$

$SH^*_n(D^*L) \underset{\text{grp}}{\cong} H^n(\Lambda L)$ (Viterbo)

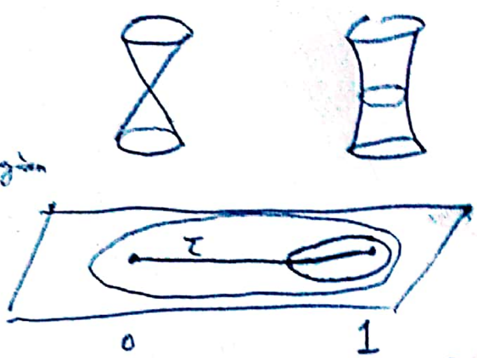
w/ these iso, $c_*: H^*(\Lambda L) \longrightarrow H^*(L)$ is in fact, the map induced by $L \hookrightarrow \Lambda L$. This map is surjective. But if $SH_*(M) \xrightarrow{c_*} \mathbb{Z}/2$ is not surj; ie is 0; also $j! \neq 0$, then the diagram doesn't commute. ↯ ▨

So if M has an exact Lag, then $SH^*(M) \neq 0$ over $\mathbb{Z}/2$ or, if L is spin, use any coeff.

Also: If $W \hookrightarrow M$; $SH^*(M)=0$, then since $SH^*(M) \longrightarrow SH^*(W)$ is a unital ring morph,
$0_M = 1_M \longmapsto 1_W = 0_W$.

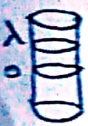
So we have $SH^*(W)$ as an obstruction to embedding.

Auroux, Wall Crossing & T-Duality: Let $M = \mathbb{C}^2 \setminus \{\underbrace{x^2+y^2}_{\overset{\shortparallel}{f}} = 1\}$



We have product (Clifford) & Chekanov tori; they are Lagrangian isotopic but not Ham iso:

Chose



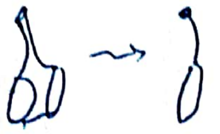
can isotope up to height λ & pass over the singularity, then come back down. But this doesn't preserve signed area & hence, isn't a Ham iso.

(They're Lag by parallel transport: fibers are symp so they intersect their ω-orthogonal complement transversally)

the tori are distinguished by the Lefschetz thimble & the vanishing cycle is removed so there is no obstruction for the tori to be exact.

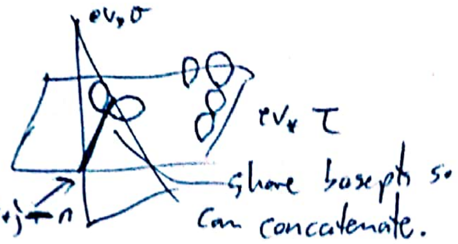
Product Struc: $SH^i(M) \otimes SH^j(M) \xrightarrow{p} SH^{i+j-n}(M)$;

, unit:



cf. loop product (Chas-Sullivan): Q - closed, smooth mfd. $\Lambda Q \xrightarrow{ev} Q$, $\gamma \longmapsto \gamma(1)$, $\sigma \in H_i(\Lambda Q)$, $\tau \in H_j(\Lambda Q)$

Gives prod $H_i(\Lambda Q) \otimes H_j(\Lambda Q) \xrightarrow{\ell} H_{i+j-n}(\Lambda Q)$.



Thm (Abbondandolo-Schwarz): Let Q be closed & spin. Then over any field, $(SH^*(T^*Q), p) \underset{\substack{\text{ring}\\ \text{iso}}}{\cong} \underbrace{(H_*(\Lambda Q), \ell)}_{\substack{\text{homotopy}\\ \text{invariant of}\\ Q}}$.

e.g.
$$\begin{tikzcd} \Omega S^n \arrow[r, hook] & \Lambda S^n \arrow[d] \\ & S^n \end{tikzcd}$$
; Aguade: $\Lambda S^n \simeq \Omega S^n \times S^n$ iff $n = 1, 3, 7$.

Lemma: $\forall n$, $H_*(\Omega S^n) = \begin{cases} \mathbb{Z}, & * = 0, n-1, 2(n-1), 3(n-1), \dots \\ 0. & \text{else} \end{cases}$

Hence, $SH^*(\underset{\overset{\shortparallel}{\mathbb{C}^*}}{T^*S^1}, \mathbb{C}) \cong \mathbb{C}[t, t^{-1}]$; observe $\operatorname{Spec} \mathbb{C}[t, t^{-1}] \cong \mathbb{C}^*$.

Conjecture: Mirror of log CY X is $\operatorname{Spec} SH^*(X)$.

SH^* also has Lie bracket & BV operator ...

Let M be a spin Liouville domain & $\Delta \hookrightarrow (M \times M, \omega \oplus (-\omega))$ the diagonal, which is Lag.

Then $SH^*(M) \cong HW^*(\Delta, \Delta)$.

Lastly: If M^{2n} is Weinstein, there are maps $HH_{*-n}(\mathcal{W}(M)) \longrightarrow SH^*(M) \longrightarrow HH^*(\mathcal{W}(M))$ that are both isomorphisms.

(Ganatra, GPS)

Last time: Viterbo functoriality; let $W \xrightarrow[\text{codim } 0]{j} M$ embedding of Liouville domains; i.e. $j^* \lambda_M = \lambda_W + dK$ & Suppose $\exists J$ s.t. its Floer traj b/w orbits of W, stay in W. Then $\exists$ a map $F_j^* : SH^*(M) \longrightarrow SH^*(W)$ which is a unital ring morphism, fitting into the diagram:

$$\begin{tikzcd} SH^*(M) \arrow[r] & SH^*(W) \\ H^{n+*}(M) \arrow[u, "c^*"] \arrow[r, "j^*"] \arrow[ur, phantom, "\circlearrowleft"] & H^{n+*}(W) \arrow[u, "c^*"'] \end{tikzcd}$$

The map is functorial wrt Liouville inclusions of this type.

Remarks: 1 every map is a ring map. The dual diagram then has a counital coalgebra map.

When we have an exact Lag $L \subset M$, we can embed $D^*L \hookrightarrow M$. Weinstein nghd thm says we can always do this symplectically but the exactness condition is needed for the embedding to be one of Liouville domains.

3. Let $L \hookrightarrow T^*N$. We then get a (dual) diagram

$$\begin{tikzcd} H^*(\Lambda L) \arrow[r, "F_j^!"] \arrow[d, shift left, "i^*"] \arrow[dr, phantom, "\circlearrowleft"] & H^*(\Lambda N) \arrow[d, shift left, "ev_0^*"] \\ H^*(L) \arrow[u, shift left, "ev_0^*"] \arrow[r, "j^!"] & H^*(N) \arrow[u, shift left, "i^*"] \end{tikzcd}$$

There is a spectral enrichment by Kragh:

$$\begin{tikzcd} \Lambda N^{-TN} \arrow[r] & \Lambda L^{-TL+\eta} \\ N^{-TN} \arrow[u] \arrow[r] & L^{-TL} \arrow[u] \end{tikzcd}$$

These are Thom spectra; view $-TL$ as a virtual bundle or as some stabilized normal bundle of $L \hookrightarrow \underbrace{S^M}_{\text{large sphere}}$

I mention this (not b/c I understand it) b/c it is used to prove:

Thm (Abouzaid-Kragh): When $n \equiv 1, 3, 5 \bmod 8$, $\exists$ a class of Lag immersions of $S^n \looparrowright T^*S^n$ in the homotopy class of the 0 section which do <u>not</u> admit embedded representatives.