Oral Exam Questions

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1 Potential Oral Questions

Before the exam, I wrote up fair questions I thought they could ask me.

1.1 Major Topic: Hamiltonian Floer theory

Questions:

1. What does it mean for a critical point to be nondegenerate in the Morse theory sense? What is a Morse function? Why are Morse functions dense in $C^\infty(M)$?

2. What does it mean for a pseudogradient field to be Morse-Smale? Relate this to stable and unstable manifolds.

3. Define the Morse complex and show that the Morse homology is independent of Morse function and Morse-Smale pseudogradient field.

4. Explain how Morse theory gives us a cell decomposition of a compact manifold.

5. State Arnold's conjecture in our setting and explain the terms. For example, what does it mean for a Hamiltonian to be nondegenerate?

6. What is the action functional and what is its differential?

7. Why do we need that $\pi_2(\omega) = 0$?

8. Derive the Floer equation $\mathcal{F}u = 0$ and describe what it means.

9. What does it mean for Morse/Floer trajectories to converge?

10. Prove the moduli space of Morse trajectories is compact.

11. Prove the moduli space of Floer trajectories is compact.

12. Define the Maslov index. Why do we need the $\langle c_1, \pi_2(M)\rangle = 0$ condition?

13. Let's fix critical points x and y and consider maps $u : \mathbb{R} \times S^1 \to M$ that decay towards x and y. Prove that for a generic (regular) Hamiltonian $H = H_0 + h$, $\sigma(u) = \mathcal{F} + \nabla_u h$ is transverse to the zero section.

14. Explain why $\mathcal{F}$ is Fredholm and compute its index along u which connects x and y. Thus, explain why the moduli space $\mathcal{M}(x, y, J, H)$ is a smooth manifold of dimension $\mu(x) - \mu(y)$.

15. Define the Floer chain complex (so define ∂ as well). What do we need to ensure that $\partial^2 = 0$?

16. Explain the gluing theorem and how to prove it.

17. Give an outline for proving the Arnold conjecture in our setting.

18. How do we show that if H is a C^2 small, autonomous Hamiltonian, then the Floer complex coincides with the Morse complex?

19. How do we show that the Floer homology is invariant under our choice of regular pairs (H, J)? (use a homotopy) How do you know it's independent of your choice of homotopy?

20. Suppose that we tried to apply the techniques in Audin and Damian to a noncompact symplectic manifold. Say, $(\mathbb{R}^2, \omega)$ and $H(x, y) = 10(x^2 + y^2)$. What goes wrong?

1.2 Minor Topic: Seiberg-Witten Theory

1. What is a $Spin^c$ structure P on a closed, orientable Riemannian 4-manifold (X, g)? What is the spin bundle?

2. Define the Dirac operator $\partial\!\!\!/_A$ twisted with a connection A on $\det P$.

3. Why is $\partial\!\!\!/_A$ elliptic? Fredholm? Self-adjoint? What is its (principal) symbol?

4. Write down the Seiberg-Witten equations and explain what they mean.

5. Describe the configuration space $\mathcal{C}(P)$ and the group of gauge changes $\mathcal{G}(P)$ and its action on $\mathcal{C}(P)$. If (A, ψ) is a solution to the SW equations and $\sigma \in \mathcal{G}(P)$, is $(A, \psi) \cdot \sigma$ a solution?

6. Describe the space $\mathcal{B}(P) = \mathcal{C}(P)/\mathcal{G}(P)$. For example, what are the singularities like?

7. Write down the elliptic complex that we need to find the dimension of $\mathcal{M}^*(P)$. Explain why it is a complex; i.e. show $D_2 \circ D_1 = 0$. Then compute the dimension $\mathcal{M}^*(P)$.

8. For an almost complex 4-manifold, why is the dimension of the moduli space zero?

9. Why is $\mathcal{M}^*(P)$ Hausdorff? Why is it compact? In fact, why are there only finitely many $Spin^c$ structures with nonempty moduli spaces, all of them compact?

10. Why is $\mathcal{M}^*(P)$ a smooth manifold? i.e. prove transversality.

11. Discuss why, when $b_2^+ > 1$, we can obtain a moduli space free of reducible solutions. i.e. describe how to perturb the SW equations.

12. How do we orient $\mathcal{M}(P)$ (no reducible solutions)?

13. Define the Seiberg-Witten invariant when $b_2^+ > 1$.

14. Explain why the invariant, when $b_2^+ > 1$, is independent of Riemannian metric and perturbation.

15. So we know the invariant doesn't depend on the Riemannian structure. Why does it depend on the smooth structure? i.e. why is this a smooth invariant and not merely a topological invariant that does not detect smooth structural differences?

16. Can you give an example of two homeomorphic but not diffeomorphic 4-manifolds using SW invariants?
17. Can you give examples of where SW invariants are not able to help us distinguish smooth structures? (e.g. S^4 due to positive scalar curvature metrics).
18. What happens when $b_2^+ = 1$? State the wall-crossing formula and how to obtain it. In particular, explain why the cobordism that we get has only one reducible solution but is not a singular point in the cobordism.
19. Discuss SW in the case of Kähler surfaces. In particular, the Dirac operator, solutions, and the invariant. Give an example of homeomorphic but not diffeomorphic surfaces.

2 Actual Orals

I'm trying to remember what questions were asked and I'm having a hard time, despite the fact that I took the orals just 3 hours ago! I think the adrenaline has something to do with this memory-loss. Still, I'll do my best and put them in the order they were asked and record a paraphrase of my answer (organized). The committee was:

- Major Advisor: Mark McLean
- Minor Advisor: John Morgan
- Impartial Judiciary: Olga Plamenevskaya

Due to some special circumstances, John used video conferencing to ask questions. We set up a TV and webcam in the 5th floor seminar room. The exam lasted 75 minutes with 40 of it being on the major topic and the rest on the minor.

2.1 Major Topic: Hamiltonian Floer theory

1. Mark: **What are the Floer groups?** They come from chain complexes which are $\mathbb{Z}_2$-vector spaces generated by critical points of an action functional. There is a grading by the Maslov index. The group comes then by taking homology.
2. John: **Hold on, what are the initial data we need?** Let's consider a closed symplectic manifold (W, ω) and a time-dependent Hamiltonian $H : W \times \mathbb{R} \to \mathbb{R}$.
3. John: **Okay, so let's talk about the chain complex again. What are the critical points you mentioned?** They are the critical points of an action functional. Let $x : S^1 \to W$ be a contractible loop; let $\mathcal{L}W$ denote the space of free contractible smooth loops. The action functional is:

$$\mathcal{A}_H : \mathcal{L}W \to \mathbb{R}; \quad \mathcal{A}_H(x) = -\int_{D^2} u^*\omega + \int_0^1 H_t(x(t))$$

   where $u : D^2 \to W$ is an extension of x. The critical points of $\mathcal{A}_H$ satisfy the Hamiltonian equation: $\dot{x}(t) = X_t(x(t))$ where these X_t are a family of Hamiltonian vector fields.
4. John: **What happens if we choose a different extension?** That would be a problem but we make the assumption of $\pi_2(\omega) = 0$. This means that the area of any S^2 mapped into W is zero. Thus, we may glue two disks along their boundary and... (John cut me off, knowing where I was going).

5. Mark and John **What is the differential?** To define it, we need to count trajectories of $\nabla \mathcal{A}_H$. So this gradient means we have chosen an almost complex structure J compatible with ω which defines a Riemannian metric that we lift to the loop space. It was the genius of Floer to realize the trajectories satisfy a PDE and by studying the PDE, we can understand the geometry: (I wrote down the Floer equation): let $u : \mathbb{R} \times S^1 \to W$ be a smooth map;

$$\mathcal{F}u = \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \nabla_u H_t = 0.$$

   and talked about how the moduli space of solutions with finite energy can be counted and that's how we define the differential. (I wrote down the definition for energy).

6. Mark and John: **What is the moduli space that we need with respect to two critical points x and y? Do all finite energy u connect critical points?** You mean $\mathcal{M}(x, y)$. **Yes, write mathematically what this space is.** They are $u \in \mathcal{M}$ such that $\lim_{s\to-\infty} u(s, t) = x(t)$ and similarly for y. And yes, all finite energy u connect critical points.

7. Mark: **What is the energy of $u \in \mathcal{M}(x, y)$?** It is $E(u) = \mathcal{A}_H(x) - \mathcal{A}_H(y)$.

8. Mark: **Why do we need the finite energy condition?** Without it, we don't have compactness and I wouldn't know how to count trajectories.

9. John: **What is the dimension of $\mathcal{M}(x, y)$?** It is the Maslov index difference $\mu(x)-\mu(y)$.

10. Olga and John: **Do we need genericity? What is it that we perturb? How do we show the invariance of the homology under these generic choices?** We do need genericity for transversality to hold and thus, obtain a smooth manifold as our moduli space. We can either perturb the almost complex structure J or the Hamiltonians; I said that the initial data has this fixed Hamiltonian so we don't perturb it. But the Arnold's conjecture, which was the original motivation of Floer theory, is about any Hamiltonian. So the authors elect to perturb H.

    To show invariance, we define a chain map Φ^Γ by looking at the moduli space of a modified Floer equation.

11. John: **Can you explicitly describe the chain map?** Yes, we need a homotopy Γ between choices of regular pairs, say (H^a, J^a) and (H^b, J^b). Φ^Γ is defined by counting trajectories which is possible because if x and y are of the same index, despite being critical points for different action functionals, $\dim \mathcal{M}^\Gamma(x, y) = \mu(x) - \mu(y) = 0$.

12. Mark: **So this is a family of regular pairs?** Yes, I got ahead of myself. The family gives a modified Floer equation which produces a moduli space $\mathcal{M}^\Gamma$. What we have are critical points of $\mathcal{A}_{H_a}$ and $\mathcal{A}_{H_b}$ and trajectories between them.

13. Mark: **Why is this a chain map?** We need to show $\partial \circ \Phi^\Gamma = \Phi^\Gamma \circ \partial$. To do that, let me draw a suggestive picture that illustrates convergence to broken trajectories. (I drew the usual picture). We have this equality here because gluing tells us that the compactified moduli space is a 1-manifold with boundary. These have an even number of boundary points and taken mod 2, is 0.

14. Olga: **We have yet to see the invariance. Can you show it?** Ah, I see. Because we've made choices of Γ. To see the homology is independent of these choices, we need to define a chain homotopy. We also need to show that when $(H^a, J^a) = (H^b, J^b)$ and $\Gamma = \mathrm{id}$, the $\Phi^\Gamma = \mathrm{id}$. We then work with three chains (so introduce a (H^c, J^c)) and show

that if we have homotopies linking every pair, then the composition of say, the one from a to b followed by b to c is equal on the homology level to the one straight from a to c. The chain homotopy will (I did not say this very clearly and went in some circles).

15. Mark: **In the case that** $(H^a, J^a) = (H^b, J^b)$ **and** $\Gamma = \mathrm{id}$**, is** $\Phi^\Gamma = \mathrm{id}$ **on the chain level or homology level? And does it equal this composition** $\Phi^{\Gamma_1} \circ \Phi^{\Gamma_2}$ **on the chain or homology level?** $\Phi^\Gamma = \mathrm{id}$ on the chain level but it equals the composition of the other two on homology.

2.2 Minor Topic: Seiberg-Witten Theory

1. John: **What are the Seiberg-Witten equations and what do the symbols mean?** They are as follows:

$$F_A^+ = q(\psi) := \psi \otimes \psi^* - \frac{|\psi|^2}{2}\,\mathrm{id}; \qquad \partial\!\!\!/_A \psi = 0.$$

   We have a fixed $Spin^c$ structure P which any closed, orientable Riemannian 4-manifold (X, g) always admits. The second equation is easy to describe; we look at the Dirac operator defined on the spin bundles; this ψ is a section, also called a spinor.

2. John: **Hold on, what are the spin bundles?** They come from representations of $Spin^c$; we can form an associated vector bundle to a principal-G bundle if we have a fixed vector space V and a linear representation $G \to GL(V)$.

3. John: **What is** $Spin(4)$**? What are the representations of** $Spin(4)$**?** $Spin(4) \cong SU(2) \times SU(2)$. To discuss the representations takes us into Clifford algebras. We naturally look at finite dimensional complex irreducible representations and ask, which of these restrict naturally to $Spin(4)$? The representations are... (I struggled awhile because what I said at first was not correct; I said we need to further restrict to each $SU(2)$. I should have said, we project. Mark stepped in to clarify John's question). Oh, there are only two representations $\rho : SU(2) \times SU(2) \to GL(2, \mathbb{C})$. Simply project on each $SU(2)$ and take the natural action.

4. John: **What is the** A **you've written in the equations?** It's a $U(1)$ connection on the associated determinant line bundle of P.

5. John: **Continue explaining what the curvature equation means**. Well, we have F_A^+ which is the self-dual curvature of A. On the RHS, we have an traceless endomorphism of $S^+(P)$. How are they equal? Well, there is a natural action of differential forms on the spin bundles because the Clifford algebra is isomorphic as a vector space to the exterior algebra of differential forms. Also, we have a special element $\omega_{\mathbb{C}}$ which squares to $+1$ so it has $+1$ and -1 eigenspaces; much like Hodge-$*$.

6. John: **So what is the moduli space? And what properties does it have?** It is the space of solutions and it has lots of desirable properties like...

7. Mark: **Wait, shouldn't we quotient by something?** Ah yes, otherwise it's an infinite dimensional space of solutions. We quotient by a group of change of gauge.

8. John: **And what is that group?** It is denoted $\mathcal{G}(P)$. It is the group of automorphisms on our bundle P which are the identity on the frame bundle.

9. John: **What do you mean by that?** Well $P \to X$ covers the frame bundle F. **You mean as in a covering space?** No. As bundles. **So what's the fiber?** Just S^1.

10. John: **What's the action of $\sigma \in \mathcal{G}(P)$ on the pair (A, ψ)?** We get induced maps $\det \sigma$ and $S^{\pm}(\sigma)$ on the line bundle and spin bundles. Then $(A, \psi) \cdot \sigma = ((\det \sigma)^* A, S^+(\sigma^{-1})\psi)$. **What is the action of $S^+(\sigma^{-1})$ on ψ?** Clifford multiplication.

11. John: **What is the dimension of $\mathcal{M}(P)$ and how do we compute it?** We look at an elliptic complex at (A, ψ):

$$0 \to W^{3,2}(X; i\mathbb{R}) \to W^{2,2}((T^*X \otimes i\mathbb{R}) \oplus S^+(P)) \to W^{1,2}((\Lambda^2_+ T^*X \otimes i\mathbb{R}) \oplus S^-(P)) \to 0.$$

The nontrivial maps in order are $D_1 = (2d, - \cdot \psi)$ which is the differential of gauge action and
$$D_2 = \begin{pmatrix} d^+ & -D_{q(\psi)} \\ \cdot \frac{1}{2}\psi & \partial\!\!\!/_A \end{pmatrix}$$
which is the differential of the Seiberg-Witten operator.

The dimension is minus the Euler characteristic of this complex which is $(c_1^2(L) - 2\chi - 3\sigma)/4$; χ is the Euler characteristic of X and σ the signature.

12. John: **What do we need to have smoothness at a point?** We need to be looking at a irreducible solution.

13. John: **That's necessary but we need more.** I'm not sure what you're getting at... Do you mean $b_2^+ > 0$? **No, we need an obstruction map to be surjective (something something) Kuranishi model (something something) the second homology of the elliptic complex...** Okay...I think you're getting at transversality. **Yes, that amounts to the map being surjective.** I see. (I'm disappointed because I know transversality but didn't understand what he was asking).

14. John: **In Donaldson theory, there are some differences to the SW equations. What do we have in the SW equations that gives us a much desired property?** The much-desired property is compactness; we obtain it by *a priori* pointwise bounds from the equations.

15. John: **Talk me through how we obtain this compactness.** It is firstly, a very strong compactness. For all the $Spin^c$ structures with non-negative dimension, we obtain compact moduli spaces and there are only finitely many of them with nontrivial SW invariant. The main tool is the Weitzenböck formula. That is
$$\partial\!\!\!/_A^2 \psi = \nabla_A^* \nabla_A \psi + \frac{\kappa}{4}\psi + \frac{F_A}{2} \cdot \psi.$$
I'm impressed you remember that. I would have to look it up. Continue. We have the LHS equals 0 as ψ is a solution for the Dirac equation; now take an inner product with ψ. We get
$$0 = |\nabla_A \psi|^2 + \frac{\kappa}{4}|\psi|^2 + \frac{|\psi|^4}{4}$$
with the last equality coming from the fact that ψ is a $+$-spinor.

16. John: **So what does this say about manifolds like S^4 or $\mathbb{C}P^2$?** Well, $|\nabla_A \psi|^2 \geq 0$ so then we have pointwise bounds from κ, the scalar curvature

$$|\psi(x)|^2 \leq -\kappa(x) \leq \max\{-\kappa(x), 0\} \leq \max_{x \in X} \max\{-\kappa(x), 0\} =: \kappa_X^-.$$

Thus, there are only reducible solutions on S^4 and $\mathbb{C}P^2$. Moreover, we can relate $|\psi|^2$ to the norm of $q(\psi)$ and thus $|F_A^+|$. Of course, $c_1^2(L) = \frac{1}{4\pi^2}(|F_A^+|^2 - |F_A^-|^2)$; the minus comes from the fact that F_A^- is antiself-dual. The scalar curvature and other topological data gives a bound on in cohomology for what c_1^2 can be; since it's discrete, we get finitely many nontrivial possibilities.

That was it. They sent me out of the room and discussed for just a few minutes before bringing me into to say I passed. John reiterated that he was impressed I remembered the Weitzenböck formula. I told him I reviewed it recently and that I still had to return his copy of Freed and Uhlenbeck's *Instantons and 4-Manifolds*.