Principal G-Bundles

Sam Auyeung

October 28, 2019

These notes were taken from a reading of John Morgan's *Introduction to Gauge Theory*, lecture notes which appear in *Gauge Theory and the Topology of 4-Manifolds*, a book in the IAS/Park City series.

1 Introduction

Classically, when studying the topology of smooth manifolds, we have a number of invariants to consider. There is the homotopy type which may sometimes be replaced by the cohomology ring. We may also consider the characteristic classes of a manifold on its vector bundles. For example, the 1st Stiefel-Whitney class determines orientability. If the manifold is almost complex, the top Chern class is the Euler class. If the manifold is a $4k$-manifold, the Pontrjagin classes are homeomorphism invariants.

In dimensions above 4, surgery theory has been amazingly successful at understanding the smooth structures and topology of manifolds. In dimensions 1,2, and 3, topology and smooth structures coincide. That is, for a given topological manifold, there is exactly one smooth structure, up to diffeomorphism. However, dimension 4 is peculiar in that neither these lower dimensional or higher dimensional techniques/facts, hold.

In dimension four, if we assume the smooth manifold M is closed, orientable, simply connected, then the only interesting homology class to consider is $H_2(M, \mathbb{Z})$, on which there is an intersection form. Knowing about the intersection form of a topological manifold gives quite a lot of information. Michael Freedman showed that, for each unimodular symmetric bilinear form q (up to isomorphism), there is a topological 4-manifold with q as its intersection form. If q is even, the manifold is unique (up to homeomorphism). If it is odd, then there are two.

It is then an interesting question to ask: what can the intersection form tell us about smooth structures? This is where gauge theory enters the picture, through the work of Sir Simon Donaldson, using the Yang-Mills functional. He showed that for a simply connected, closed, smooth 4-manifold, if its intersection form is definite, then the form is diagonalizable over the **integers**. This shows, for example, that there are topological 4-manifolds which do not admit any smooth structure, such as the E_8 manifold.

2 Principal G-Bundles and Connections

Usually, when considering a principal bundle, we have a structure Lie group G and a base B which is a topological space. We may assume the base to be a simplicial or CW complex and apply homotopy theory. Or we assume B is smooth and the total space of the bundle to also be a smooth manifold.

2.1 Principal Bundles

Definition 2.1. *A (right) principal G-bundle is a triple (P, B, π) where $\pi : P \to B$ is a map. There is a continuous, free right action $P \times G \to P$ with respect to which π is invariant. That is, $\pi(pg) = \pi(p)$. Thus, π induces a homeomorphism between the quotient space of this action and B. There is also an open covering $\{U_\alpha\}$ of B and homeomorphisms φ_α which give us the following commutative diagram:*

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[rr, "\varphi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times G \arrow[dl, "p_1"] \\ & U_\alpha & \end{tikzcd}$$

*p_1 is projection onto the first factor and φ_α is G-**equivariant** with respect to the right G-action. This means φ_α commutes with the right G-action: $\varphi_\alpha(pg) = \varphi_\alpha(p)g$. The φ_α are called **local trivializations**.*

If P and B are smooth manifolds, the G action is smooth, and π is a smooth submersion, then the φ_α are automatically diffeomorphisms. An **isomorphism** of G-bundles over the same base is a homeomorphism between their total spaces which is G-equivariant and commutes with the projections to the base. A map between G-bundles with possibly different bases is a G-equivariant map between total spaces and is an isomorphism on each fiber. Such a map also induces a map on the base spaces.

Note that these local trivializations give rise to local sections $\sigma : U_\alpha \to P$. Choose a $g \in G$ and let $x \in U_\alpha$. Then since φ_α is a diffeomorphism and $\varphi_\alpha^{-1}(x, g) \in \pi^{-1}(U_\alpha)$, this gives a way to define our local section σ. Note that the converse is also true. If we start with a local section, because of the G-invariance of π, it gives us a local trivialization. We simply translate the image of the section by G action (which is free). This gives us a result.

Proposition 2.2. *Let $P \to B$ be a smooth G-bundle. Then $P \cong B \times G$ if and only if there is a smooth global section $\sigma : B \to P$.*

2.2 Examples

Example 2.3. Let M be a smooth n-manifold; then the frame bundle F for its tangent bundle is constructed as follows. Let $x \in M$. The fiber over x is the space of all bases for T_xM. This space is acted on by $GL(n, \mathbb{R})$ and in fact, this shows us that F is a $GL(n, \mathbb{R})$ bundle. If (M, g) is an orientable Riemannian manifold, we can take oriented orthonormal bases instead and form a $SO(n)$-bundle instead.

Example 2.4. Consider $\mathbb{C}P^n$; we can form the tautological bundle in the following way. A point $z \in \mathbb{C}P^n$ corresponds to a complex line in $\mathbb{C}^{n+1}$ through the origin. Each complex line contains a copy of S^1 which can be obtained simply by looking at the unit vectors in the line. Therefore, we can also consider all the unit length vectors of $\mathbb{C}^{n+1}$ which is the unit sphere S^{2n+1}. In coordinates, $S^{2n+1} = \{(z_1, ..., z_{n+1}) : |z_1|^2 + ... + |z_{n+1}|^2 = 1\}$. S^1 acts by left-multiplication and clearly preserves the norm. This data determine a principal S^1-bundle, the tautological bundle over $\mathbb{C}P^n$

By the way, if we think of S^{2n+1} inside of $\mathbb{R}^{2n+2}$ with coordinates $p = (x_1, y_1, ..., x_{n+1}, y_{n+1})$, then the 1-form

$$\frac{1}{||p||} \sum_{j=1}^{n+1} y_j dx_j - x_j dy_j$$

is dual to the vector field which rotates each plane spanned by $\{x_j, y_j\}$ counterclockwise. We normalize so that the rotation is uniform.

2.3 Transition Functions

Let $\pi : P \to B$ be a principal G-bundle with an open covering $\{U_\alpha\}$ on B and local trivializations φ_α defined by $\varphi_\alpha(p) = (\pi(p), g_\alpha(p))$. Then, from these data are **transition functions** $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ defined by $g_\alpha(p) = g_{\alpha\beta}(\pi(p)) \cdot g_\beta(p)$ for all $p \in \pi^{-1}(U_\alpha \cap U_\beta)$. These satisfy the **cocycle** condition: $g_{\alpha\beta} \cdot g_{\beta\gamma} = g_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$.

Then $g_{\alpha\alpha} = \mathrm{id}$ and $g_{\beta\alpha} = g_{\alpha\beta}^{-1}$. If we have another set of trivializations on $\{U_\alpha\}$ which gives rise to g'_α, there are maps $h_\alpha : U_\alpha \to G$ such that $g'_\alpha(p) = h_\alpha(\pi(p)) \cdot g_\alpha(p)$. Then, the new transition functions are related to the old ones by $g'_{\alpha\beta} = h_\alpha \cdot g_{\alpha\beta} \cdot h_\beta^{-1}$ on $U_\alpha \cap U_\beta$. In this case, we say the two transition cocycles differ by a **coboundary**.

With this in mind, we have a different way to describe a principal G-bundle over a base B. Suppose we have an open covering $\{U_\alpha\}$ of B and functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ which satisfy cocyle conditions. We form a space P by taking the quotient space of

$$\coprod_\alpha U_\alpha \times G$$

under the equivalence relation $(u, g) \in U_\alpha \times G$ is identified with $(u, g_{\beta\alpha}(u) \cdot g)$ for any $u \in U_\alpha \cap U_\beta$. We are gluing the pieces $U_\alpha \times G$ together.

2.4 Constructing Bundles

There is the usual construction of bundles by pullback. If $f : A \to B$ is a continuous map and $P \to B$ is a principal G-bundle, then $f^*P \to A$ is a principal G-bundle. The total space is the fibered product which means $f^*P = \{(a, p) : f(a) = \pi(p)\} \subset A \times P$. That is, we make a fiber over a by taking the fiber over $f(a)$.

We can also form associated vector bundles from principal bundles. Let $\rho : G \to GL(V)$ be a linear representation of G on some vector space V. This gives us a natural left action of G on V. Thus, we can form a total space $P \times_G V$ which is the quotient of $P \times V$ under the equivalence $(p \cdot g, v) \sim (p, g \cdot v)$. This gives us a vector bundle.

Example 2.5. An example is the **adjoint** representation of G on its Lie algebra $\mathfrak{g}$: $G \times \mathfrak{g} \to \mathfrak{g}$ given by $(g, X) \mapsto g \cdot X \cdot g^{-1}$. The vector bundle associated to $P \to B$ and this representation is denoted $\mathrm{ad}\, P$, the adjoint bundle of P.

A major result is that we may classify all principal G-bundles up to isomorphism by considering pullback bundles from a universal bundle. For a Lie group G, there is a classifying space BG and a universal, contractible G bundle $EG \to BG$. Let X be a smooth manifold and $f : X \to BG$ a smooth map. Then, f^*EG gives a smooth principal G-bundle over X. In fact, **all** principal G-bundles over X are isomorphic to a pullback bundle f^*EG, coming from some map f. The isomorphism classes of principal G-bundles are in 1-1 correspondence with homotopy classes of maps $f : X \to BG$.

2.5 Connections on Smooth Principal Bundles

Let $\pi : P \to B$ be a smooth principal G-bundle over an n-manifold. A connection for $P \to B$ is an infinitesimal version of an equivariant family of cross sections.

Definition 2.6. *A **connection** on this bundle $\pi : P \to B$ is a rank n distribution H of the tangent bundle TP which is **horizontal** in the sense that the restriction of $d\pi$ to each plane in H is an isomorphism onto the corresponding tangent space to B. H is also invariant under G action.*

Such a distribution is a family of complementary subspaces to the subbundle TP^v of tangents along the fibers. The vectors of TP^v are also called vertical tangents as they point in the direction of each fiber which are all diffeomorphic to G. Thus, a distribution gives us an isomorphism $T_pP \cong T_pP^v \oplus T_{\pi(p)}B$.

Lemma 2.7. *Suppose H is a connection for $P \to B$. Let $\gamma : [0,1] \to B$ be a smooth path and $e \in \pi^{-1}(\gamma(0))$. Then there is a* ***unique*** *path $\tilde{\gamma} : [0,1] \to P$ such that $\tilde{\gamma}(0) = e$, $\pi \circ \tilde{\gamma} = \gamma$, and $\tilde{\gamma}'(t)$ is contained in the horizontal space $H_{\tilde{\gamma}'(t)}$.*

The proof amounts to considering the pullback bundle γ^*P over the $[0,1]$ and looking for existence and uniqueness of integral curves. This becomes an application of the ODE theorem.

Corollary 2.8. *Given a smooth curve in the base γ from b_0 to b_1, a connection determines an isomorphism between the fibers $\pi^{-1}(b_0) \to \pi^{-1}(b_1)$ which is equivariant with respect to the G-actions on these fibers.*

This is the reason for the name connection. They give a way to connect distinct fibers.

2.6 Connection 1-Forms

There is a unique 1-form $\omega_{MC} \in \Omega^1(G, \mathfrak{g})$ which is invariant under left multiplication by G and at the identity element e of G, it is the linear identity map $T_eG \to \mathfrak{g}$. This form is called the Mauer-Cartan form and is often denoted $g^{-1}dg$. Its value on $\tau \in T_gG$ is equal to $g^{-1} \cdot \tau \in T_eG = \mathfrak{g}$.

Lemma 2.9. *A connection on a smooth principal bundle $\pi : P \to B$ is equivalent to a differential 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ with the following properties:*

- *Under right multiplication by G, ω transforms via the adjoint representation of G on $\mathfrak{g}$: $\omega_{pg}(\tau \cdot g) = g^{-1}\omega_p(\tau) \cdot g$, for any $p \in P$, any $\tau \in T_pP$, and any $g \in G$.*
- *For any $p \in P$, consider the embedding $R_p : G \to P$, $g \mapsto pg$; this embedding is another way to see G is a (vertical) fiber. Then the pullback $R_p^*\omega = \omega_{MC}$.*

Proof. Suppose ω has the two properties. Then let H_p be the kernel of $\omega_p : T_pP \to \mathfrak{g}$. Since R_p is an embedding of G into a fiber (which are thought of as vertical), we get an isomorphism $dR_p : T_gG \to T_{pg}P^v$.

The second property of ω says that restriction to the vertical tangents gives an isomorphism $T_pP^v \to \mathfrak{g}$. But locally, in a neighborhood $U \subset B$, P looks like $U \times G$. Thus, $H_p = \ker \omega_p$ means it is projected isomorphically to the tangent spaces to B.

We have only to check that H is G-invariant. But the first property of ω shows that the kernel of ω is invariant: the adjoint representation is linear and so it does not affect the kernel.

Conversely, given a connection H, define $\omega_p : T_pP \to \mathfrak{g}$ to be the following composition $T_pP \xrightarrow{pr} T_pP^v \xrightarrow{(dR_p)^{-1}} \mathfrak{g}$ where the first map is linear projection with kernel H_p. Let's check that $\omega_{pg}(\tau \cdot g) = g^{-1}\omega_p(\tau) \cdot g$. Since H is G-invariant, if we set H_p as the kernel of the ω_p, then it is clear that if $\tau \in \ker \omega_p$, then $\tau \cdot g \in \ker \omega_{pg}$. And since H_p is complementary to T_pP^v, then if $\tau \in T_pP^v$, $\tau \cdot g \in T_{pg}P^v$. Thus, when $\tau \in T_pP^v$, the projection in ω_p is the identity map.

Putting this together, we know the following: $\omega_{pg}(\tau \cdot g) = (dR_{pg})^{-1}(\tau \cdot g)$ and $\omega_p(\tau) = (dR_p)^{-1}(\tau)$. Thus, we just need to show

$$(dR_{pg})^{-1}(\tau \cdot g) = g^{-1}(dR_p)^{-1}(\tau) \cdot g \Longleftrightarrow g \cdot (dR_{pg})^{-1}(\tau \cdot g) = (dR_p)^{-1}(\tau) \cdot g;$$

i.e. R_p and G-action commute. But they do. To show that ω satisfies the second condition, if $\tau \in H_p$, then its in the kernel. So consider $\tau \in T_pP^v$, the complement. Then pr is the identity and so $\omega_p(\tau) = (dR_p)^{-1}(\tau)$. Now let $\tau \in T_gG$ which will be sent to a vertical fiber. Then $(R_p^*\omega)_g(\tau) = \omega_{pg}((dR_p)_g(\tau)) = (dR_{pg})^{-1} \circ (dR_p)_g(\tau) = g^{-1} \cdot \tau$. This is the Mauer-Cartan form. $\square$

When thinking of a connection A with connection 1-form ω, we can discuss its covariant derivative ∇_A on an associated vector bundle $W \to B$. This vector bundle is associated to the principle G-bundle $\pi : P \to B$ via a linear representation of G on some V which serves as the fiber for W.

Let s be a section of $W \to B$. Locally, $s(b) = [p(b), v(b)]$ where p and v are smooth functions of b with values in P and V respectively. p is a local section of $P \to B$. Then $\nabla_A(s)(b)$ evaluated on a tangent vector $\tau_b \in T_bB$ is

$$[p(b), \omega_b(De_b(\tau_b)) + Dv_b(\tau_b)]$$

if $p(b)$ is horizontal in the τ_b direction (we can always choose this), then the above simplifies to $[p(b), \partial v/\partial\tau_b]$. This satisfies the Leibniz rule.

Then, **parallel transport** is a section s such that $\nabla_A s = 0$. It exists and is unique once we specify some initial conditions.

2.7 Curvature

Above, when we looked at parallel transport, we have the existence and uniqueness of ODEs at our disposal. Parallel transport was essentially an integral curve. However, for higher dimensions, if the horizontal distribution that is a connection has rank > 1, then, there may be an obstruction to the distribution being integrable. Curvature may be thought of as this obstruction.

Fix a point $b \in B$ and two linearly independent vectors t_1, t_2. In local coordinates $(x_1, ..., x_n)$ such that $(\partial/\partial x_i)|_0 = t_i$ for $i = 1, 2$, we can consider a rectangle $[0, \epsilon]^2$ in the (x_1, x_2)-subspace. Lifting the four sides of the rectangle in a counterclockwise fashion gives us some path in P. Say that b lifts to $p \in P$. The end point might not coincide with the starting point but it will equal $p \cdot g$ for some unique $g = g(\epsilon) \in G$. For small $\epsilon > 0$, this g is close to the identity so we can consider the element:

$$K_A(\epsilon) = -\frac{\log(g(\epsilon))}{\epsilon^2}.$$

Lemma 2.10. *The element in g given by $K_A(p, t_1, t_2) = \lim_{\epsilon\to 0} K_A(\epsilon)$ depends only on p, t_1, t_2. Moreover, $[p, K_A(e, t_1, t_2)] \in ad\, P$ depends only on t_1, t_2 and is bilinear and skew-symmetric in these variables. We obtain it by evaluating a 2-form on B with values in $ad\, P$. Denote this as F_A, on (t_1, t_2).*

This F_A is the **curvature** of A. If we're in the case of vector bundles ($G = GL(n, \mathbb{R})$), these comments basically tell us that a flat connection A; i.e. $F_A = 0$ amounts to Fröbenius' integrability theorem. One would see some Lie bracket condition appearing: $[X, Y] = 0$ for all spanning vector fields X, Y of the distribution.

We can also relate the curvature $F_A \in \Omega^2(B, \mathrm{ad}\, P)$ to the connection 1-form $\omega_A \in \omega^1(P, \mathfrak{g})$ by using $\pi : P \to B$. If we have $\eta \in \Omega^1(B, \mathrm{ad}\, P)$, then $\eta \wedge \eta$ will be defined by $\eta \wedge \eta(v, w) = \frac{1}{2}[\eta(v), \eta(w)]$ where $[,] : \mathrm{ad}\, P \otimes \mathrm{ad}\, P \to \mathrm{ad}\, P$ is the Lie bracket. Then, the 2-form $\pi^* F_A = d\omega_A + \omega_A \wedge \omega_A$. Observe that if $G = S^1$, then $\omega \wedge \omega = 0$ because the Lie bracket is trivial, seeing that the Lie algebra is $i\mathbb{R}$ which is abelian. This means that a connection of a $U(1)$-bundle has curvature $F_A = d\omega_A$; the flat connections are precisely the ones with closed connection 1-forms.