A Fun Proof that $\sum \frac{1}{n}$ Diverges

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Let's suppose that

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

converges. Let $f_n(x) = \frac{1}{n}\chi_{[0,n]}(x)$ be a scaled characteristic function and

$$f(x) := \sum_{n=1}^{\infty} \frac{1}{n}\chi_{[n-1,n]}(x).$$

Observe that on $[0, n]$, $f_n = 1/n$ while $f \geq 1/n$. So f_n is bounded by f. Also, by assumption,

$$\int_{\mathbb{R}} f(x)\, dx = \sum \frac{1}{n} < \infty.$$

So f is in $L^+ \cap L^1$. We may therefore, apply the Dominated Convergence Theorem. It says that, since $f_n \to 0$ almost everywhere, then

$$0 = \int \inf f_n = \lim \int f_n = 1.$$

That's clearly a contradiction. $\square$