A Smattering of Quantum Field Theory

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1 From QM to QFT

What is QFT? In some sense, it is replacing particles with fields. What is a particle and what is a field? Mathematically, a particle is a 0 + 1 object; has no spatial dimensions but travels in 1 time dimension. So it's a map $x : \mathbb{R} \to M$, M is some target manifold. A field is usually 3+1 object so it's a map $X : \mathbb{R}^4 \to M$.

Why make our lives more difficult in doing this in the first place? Well, the difference between QM and QFT is that it brings special relativity into account. We see that the difference between classical mechanics and special relativity is that in CM, we basically assume that particles interact instantaneously (we assume they're close enough together that forces take effect immediately). However, in SR, things do not interact immediately. And so there's a theorem that basically says all particles in SR are free particles as they don't interact at a distance. Thus, well before QFT, physicists had replaced particles with fields.

If we try to take the Klein-Gordon equation of classical field theory that one sees in relativistic settings and try to quantize it in the QM way, we will run into a problem. The quantization process in QM which takes a wavefunction ψ and gives us a probability measure $|\psi|^2 d\mu$ does not give us a probability measure in this setting.

In some more details, lets recall the QM process. If we let $\rho = |\psi|^2$, $j = \frac{1}{2m}(\bar{\psi} P \psi - \psi P \bar{\psi})$ (probability current) where P is the momentum operator, then we get the **continuity equation**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0.$$

From this, it follows using Stokes theorem that probability is conserved over time:

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^3} \rho(t, x) d^3x = 0$$

In relativistic settings, we have additionally the following energy-momentum relation: $E^2 - p^2 = m^2$ ($p \in \mathbb{R}^4$). If we quantize in the same way as in QM, we'll obtain the equation $\hbar^2 \Box \psi + m^2 \psi = 0$. $\Box = \partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \Delta$ is the d'Alembertian.

One might try to produce ρ and j in the same way as before. However, since the Cauchy data for the Klein-Gordon equation are independent, $\rho(0, x)$ can take positive and negative values and, therefore, cannot be used as density of a probability measure on $\mathbb{R}^3$.

Thus the interpretation of the Klein-Gordon equation as a relativistic analog of a single particle Schrödinger equation with the wave function ψ should be abandoned. Moreover, if $\psi(x)$ takes only real values then $\rho(x)$ vanishes identically. Yet another problem with this interpretation is that from we have $E = \pm\sqrt{p^2 + m^2}$, so that solutions to the Klein-Gordon equations contain negative energy terms as well as positive energy one.

However, as was said above for special relativity, we can replace particles with fields and things will work out better for us.

2 Stone-von Neumann and Vacuum Vectors

Now, fields have infinitely many degrees of freedom. This presents a slight but surmountable complication for us in the following way: in QFT, we have creation and annihilation operators just as in QM. We may hope to find some standard irreducible representation, just as in QM, using Stone-von Neumann theorem. In QM, the finite degrees of freedom gives us this theorem:

Theorem: Every irreducible representation of canonical commutation relations are unitary equivalent.

I think the proof may rely on having a vacuum vector (vacuum state) which we get for free when there are finitely many degrees of freedom.

In the situation with infinitely many degrees of freedom, we do not have, for free, a vacuum vector. And without it, we find that the Stone-von Neumann theorem as stated, fails. However, it is a theorem that there does, in fact, exist a vacuum vector. And thus, we do have a modified Stone-von Neumann theorem:

Theorem: Every irreducible representation of canonical commutation relations with a vacuum vector is unitary equivalent to the Fock space representation.

3 Fock Space

The Fock space for a Hilbert space H is given as follows. Let $\mathcal{H}_n = \frac{1}{n!}H^{\otimes n}$, $\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$. The symmetric group Sym_n acts on $\mathcal{H}_n$ in the obvious way. Let S_n be orthogonal projection of $\mathcal{H}_n$ onto the symmetric subspace of $H^{\otimes n}$.

Let $\mathcal{F}_n = S_n \mathcal{H}_n$. Let $\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}_n$. This is called the bosonic Fock space. Replacing Sym with Alt gives the fermionic Fock space.

4 Free Quantum Field

Let φ be the free quantum field. I think its key feature is that $\varphi(x) = \varphi^+(x) + \varphi^-(x)$, one creates and the other annhilates at point x. It is also a self-adjoint operator (well, an operator-valued distribution).

φ satisfies a commutativity relation: let $x, y \in \mathbb{R}^3$. Then $[\varphi(x), \varphi(y)] = -iD(x-y)\,\mathrm{id}$. This D is called the Pauli-Jordan function and satisfies the Klein-Gordon equation. When $|x-y|^2 < 0$ (x and y are spacelike separated, so $x-y$ is outside the light cone), $D(x-y) = 0$. So in that case, $[\varphi(x), \varphi(y)] = 0$. This property is called **causality** or **local commutativity**. We don't want space like stuff to influence the past nor future so they should commute. The proof uses the fact that D is invariant under Lorentz transformations.

5 Correlation Functions

For any operator A in the Fock space $\mathcal{F}$ its vacuum expectation value is the inner product $(A\Phi, \Phi)$. The linear span of vectors $\mathbf{a}^\dagger(\mathbf{f_1}), ...\mathbf{a}^\dagger(\mathbf{f_n})$ (creation operators) are dense in $\mathcal{F}$. So to know all matrix elements fo the quantum field φ, we just need to know the vacuum expectation values of the product $\varphi(x_1)...\varphi(x_n)$. These are the n-point correlation functions of the quantum field. Put another way, the correlation function gives the statistical average of a field at multiple spacetime points. In axiomatic QFT, these are called Wightman functions W_n.

The 2-point correlation function, in particular, is the correlation between the field at one point and another. We may think of this as the probability of an excitation at one place (a particle), moving to the other. We call this a propagator.

An interesting trick for computing W_2: complexify Minkowski space: $\mathbb{R}^4 = \mathbb{R} \oplus \mathbb{R}^3$ $(+---)$ to $\mathbb{C}^4$. Inside of $\mathbb{C}^4$ sits $i\mathbb{R} \oplus \mathbb{R}^3$ $(----)$ which is Euclidean. Complexification doesn't care about a real form so in this way, we can somehow go back and forth between Minkowski and Euclidean space. The 2-point correlation function that we care about has an analytic continuation to this larger domain $\mathbb{C}^4$ and so we can naturally shift from Minkowski to Euclidean space. Why do this? Well, physically, there is no difference but mathematically, the computations in Euclidean space are easier.

6 Feynman Path Integral

One thing I still do not really understand well is what the point of Feynman Path Integrals are. It seems that since each path has some probability, one should integrate over all paths. So you take the action $S : \mathcal{P}M \to \mathbb{R}$; its critical points are the classical solutions to the equations of motion. You then integrate:

$$\int_{\mathcal{P}M} e^{iS/\hbar}\, \mathcal{D}\mathbf{x}$$

$\mathcal{D}\mathbf{x}$ is the measure for the path space. Here, the point of multiplying by $i/\hbar$ and then exponentiating is to give each path the same *weight* but different *phase*. In this way, I think the "improbable" paths somehow cancel out (like interference). The critical points now should be almost the classical trajectories with some quantum corrections.

An important point is that we can get a **generating function** from the path integral via Taylor expansions or something. The generating function is perhaps not really a physical object so much as a mathematical object one which satisfies certain derivative conditions. The derivatives of it with respect to certain parameters gives a certain observable of the system. It's the QFT analog of a **partition function** (which appear in statistical mechanics).

7 Supersymmetric QFT

I think the basic idea of Supersymmetry is that bosons and fermions can be interchanged (in the Fock space). This has never been empirically observed. But mathematically, it's quite interesting.

It basically allows us to reduce complicated integrals over infinite dimensional spaces via localization. This just means we view the integrand as something like an exact form (in the sense of differential topology) and thus, the integral localizes to the critical points: all the interesting behavior is at critical points. Thus, the infinite dimensional space reduces to finite dimensional ones, usually moduli spaces of instantons or algebraic curves.

7.1 Donaldson Theory and Seiberg-Witten Theory

Thus, this is an application of QFT with supersymmetry to things like Donaldson or Gromov-Witten theory. Indeed, when Donaldson invariants first arrived on the scene, mathematicians worked very hard to compute them. In 1994, Kronheimer and Mrowka showed that one can theoretically compute all the Donaldson invariants by just looking at a finite set of data.

However, this breakthrough was overshadowed a few months later by Witten's work. Witten found a way to realize the Donaldson invariants as expectation values of certain observables within a TQFT. I imagine Witten looked for some generating function whose coefficients are

Donaldson invariants. This assigning of expectation values to observables via Feynman path integrals is a "quantization" process and the "topological" part of TQFT just means that the theory is independent of the Riemannian metric chosen at the start.

Within the integral, there is some number e called a coupling constant. Witten conjectured a sort of duality for when $e \to 0$ versus when $e \to \infty$.

Duality in Witten's TQFT

$e \to 0$	$e \to \infty$
Weak coupling	Strong coupling
Ultraviolet	Infrared
Magnetic	Electric
Perturbative	Nonperturbative
Donaldson invariants	Seiberg-Witten invariants

In the fall of 1994, Witten had not yet developed a way of computing the invariants on the right but soon after, Seiberg and Witten discovered how to do so. This caused quite a frenzy in the mathematics community and it seems now that people tend to work with SW invariants rather than Donaldson invariants.

7.2 Kontsevich

Another example of the relationship between QFT and math. Clifford Taubes wrote about some of the work of Maxim Kontsevich. "Kontsevich presented a proof of a conjecture of Witten to the effect that a certain, natural formal power series whose coefficients are intersection numbers of moduli spaces of complex curves satisfies the Korteweg-de Vries hierarchy of ordinary, differential equations."

I think the natural formal power series is something like a generating function. The incredible thing is that Kontsevich gives an explicit combinatorial description of the intersection numbers of the moduli spaces $\mathcal{M}_{g,n}$.