Examples of Lie Groups in Geometry and Topology

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1 Lie Groups

Lemma 1.1. *$SU(n)$ is simply connected.*

Proof. It is not hard to show that $SU(n+1)$ acts transitively on $S^{2n+1} \subset \mathbb{C}^{n+1}$. Also, if we include $SU(n)$ into $SU(n+1)$ by sending A to a matrix with A in the upper left block, a 1 in the lower right corner and $0's$ elsewhere, then we see that $SU(n)$ stabilizes $\vec{x} = (0, ...0, 1) \in S^{2n+1}$. Thus, we have a fibration and a long exact sequence of homotopy groups:

$$\begin{tikzcd} SU(n) \arrow[r, hook] & SU(n+1) \arrow[d] \\ & S^{2n+1} \end{tikzcd}$$

Since $n \geq 1$ and $\pi_1(S^{2n+1}) = \pi_2(S^{2n+1}) = 0$, we get the following LES:

$$... \to \pi_2(S^{2n+1}) = 0 \to \pi_1(SU(n)) \to \pi_1(SU(n+1)) \to 0 = \pi_1(S^{2n+1}) \to ...$$

So $\pi_1(SU(n)) \cong \pi_1(SU(n+1)$. In particular, $SU(1) = \{1\}$ which is simply connected. Thus, $SU(n)$ is simply connected for all $n \geq 1$. □

Proposition 1.2. *The map* $\det : U(n) \to S^1$ *induces an isomorphism on fundamental groups.*

Proof. We get a fibration by considering the kernel of det:

$$\begin{tikzcd} SU(n) \arrow[r, hook] & U(n) \arrow[d] \\ & S^1 \end{tikzcd}$$

Once again, we get a LES:

$$... \to \pi_1(SU(n)) = 0 \to \pi_1(U(n)) \to^{\det_*} \pi_1(S^1) \to 0 = \pi_0(SU(n)) \to ...$$

Hence, $\det_* : \pi_1(U(n)) \to \pi_1(S^1)$ is an isomorphism. □

Proposition 1.3. $\pi_2(SU(n)) = 0$.

Proof. Using the long exact sequence in the lemma above, we have

$$... \longrightarrow \pi_3(S^{2n+1}) \longrightarrow \pi_2(SU(n)) \longrightarrow \pi_2(SU(n+1)) \longrightarrow \pi_2(S^{2n+1}) \longrightarrow ...$$

When $n = 1$, $SU(1) = \{1\}$ so π_2 is trivial. When $n \geq 2$, we have that $\pi_3(S^{2n+1}) = \pi_2(S^{2n+1}) = 0$ (the lowest is $\pi_3(S^5) = 0$). Thus, for $n \geq 2$, $\pi_2(SU(n)) \cong \pi_2(SU(n+1))$. But $SU(2) = S^3$ and $\pi_2(S^3) = 0$. Thus, $\pi_2(SU(n)) = 0$. □

Remark: In fact, Milnor, in his *Morse Theory* book shows that if G is a connected Lie group, $\pi_2(G) = 0$. The proof first relies on a theorem of Iwasawa: every connected Lie group deformation retracts to its maximal compact Lie subgroup. It then considers a fibration $\Omega G \to PG \to G$. Here, PG is the path space of G with paths starting at the identity $e \in G$ and ΩG is the loop space based at e. The long exact sequence shows that $\pi_k(G) \cong \pi_{k-1}(\Omega G)$.

Equip G with a biinvariant metric (which exists since G is compact). Then, approximate the space ΩG by a nice (open) subset S of $G \times \dots \times G$ by approximating paths by broken geodesics. Short enough geodesics are uniquely defined by their end points, so the ends points of the broken geodesics correspond to the points in S. It is a fact that computing low $\pi_k(\Omega G)$ is the same as computing those of S.

Now, consider the energy functional E on S defined by integrating $|\gamma|^2$ along the entire curve γ. This is a Morse function and the critical points are precisely the closed geodesics. The index of E at a geodesic γ is, by the Morse Index Lemma, the same as the index of γ as a geodesic in G (the index of a geodesic is defined by an analog to the Hessian for the energy functional). Now, geodesics on a Lie group are very easy to work with - it's straight forward to show that the conjugate points of any geodesic have even index.

But this implies that the index at all critical points is even. And this implies that S has the homotopy type of a CW complex with only even cells. It follows immediately that $\pi_1(S) = 0$ and that $H_2(S)$ is free ($H_2(S) = \mathbb{Z}^t$ for some t). Quoting the Hurewicz theorem, this implies $\pi_2(S)$ is $\mathbb{Z}^t$.

By the above comments, this gives us both $\pi_1(\Omega G) = 0$ and $\pi_2(\Omega G) = \mathbb{Z}^t$, from which it follows that $\pi_2(G) = 0$ and $\pi_3(G) = \mathbb{Z}^t$.

Incidentally, the number t can be computed as follows. The universal cover $\tilde{G}$ of G is a Lie group in a natural way. It is isomorphic (as a manifold) to a product $H \times \mathbb{R}^n$ where H is a compact simply connected group. H splits isomorphically as a product into pieces (all of which have been classified). The number of such pieces is t.

Proposition 1.4. *Let G be a connected Lie group group. Then $\pi_1(G)$ is abelian.*

Proof. Let $p : \tilde{G} \to G$ be the universal cover of G. Then p is surjective and a Lie group morphism. Thus, the kernel of p is $p^{-1}(e) \cong \pi_1(G)$. Being the kernel, it is a normal subgroup and since G is a manifold, $\pi_1(G)$ is discrete (and must be for a covering to be evenly covering).

Lemma 1.5. *Let G be a connected **topological** group with discrete normal subgroup N. Then N is in the center of G. A nontrivial closed normal subgroup of a connected simple **Lie** group G must also be in the center.*

Proof. Fix $x \in N$. Then consider the continuous adjoint map (conjugation): $G \to N$ defined by $g \mapsto gxg^{-1}$. The map does land in N because N is normal. This is not a morphism but note that $e \mapsto x$. Since N is discrete, G is connected, and this map is continuous, all of G maps to x. So $gxg^{-1} = x$ for all $g \in G$ means $gx = xg$ for all G. But this holds for all $x \in N$. Therefore, elements of N commute with all elements of G and so $N \subset Z(G)$.

If G is a connected **Lie** group and N is a closed normal subgroup, then N is a Lie subgroup, and so its Lie algebra $\mathfrak{n}$ is an ideal of the Lie algebra $\mathfrak{g}$. If $\mathfrak{g}$ is simple, then all its ideals are 0 which means $\mathfrak{n} = 0$, and N must in fact be 0-dimensional and discrete. Then applying the previous argument, N must in fact be central. □

$\pi_1(G)$ is discrete and normal in $\tilde{G}$ which is connected. By this lemma, $\pi_1(G) \subset Z(\tilde{G})$ and is abelian. □

Here is an alternative proof:

Proof. Let G be a connected Lie group. Consider the fibration of the classifying space with the total space:

$$\begin{tikzcd} G \arrow[r, hook] & EG \arrow[d] \\ & BG \end{tikzcd}$$

We get a long exact sequence of homotopy groups. Recall that EG is contractible. Thus, we have

$$... \to \pi_2(EG) = 0 \to \pi_2(BG) \to \pi_1(G) \to 0 = \pi_1(EG) \to ...$$

So $\pi_2(BG) \cong \pi_1(G)$; since π_2 is abelian, then $\pi_1(G)$ is abelian. $\square$

There is also a third proof which is more general.

Lemma 1.6 (Eckmann-Hilton)**.** *: Let S be a set with two binary operations $\circ$ and $*$ which both have units $1_\circ$ and 1_*. Moreover, suppose that for all elements $a, b, c, d \in S$, $(a \circ b) * (c \circ d) = (a * c) \circ (b * d)$. Note that b and c have exchanged places. We'll say that this equation means that first operation is a homomorphism with respect to the second operation. Then the following hold:*

1. *The second operation is a homomorphism with respect to the first.*
2. $1_\circ = i_*$.
3. *$a \circ b = a * b$ for all $a, b \in S$. So the operations are the same.*
4. *The operation is commutative.*
5. *The operation is associative.*

Proof. (1) follows just be reading the equation from right to left. For (2), observe that $1_\circ = 1_\circ \circ 1_\circ = (1_* * 1_\circ) \circ (1_\circ * 1_*) = (1_* \circ 1_\circ) * (1_\circ \circ 1_*) = 1 * *1_* = 1_*$. Let's call the shared unit 1. For (3), $a * b = (a \circ 1) * (1 \circ b) = (a * 1) \circ (1 * b) = a \circ b$. From now on, let's just use ab to denote the binary operation on a and b.

For (4), $ab = (1a)(b1) = (1b)(a1) = ba$. Lastly, for (5), $(ab)c = (ab)(1c) = (a1)(bc) = a(bc)$. $\square$

Proposition 1.7. *Let G be a topological group with identity denoted by e. Then $\pi_1(G, e)$ is abelian.*

Proof. Observe that $\pi_1(G, e)$ has **two** operations because G is a topological group. There's the concatenation of paths which we have for any kind of fundamental group. And we have pointwise multiplication of paths because of the multiplication from G. That is, take loops α, β and define $(\alpha \cdot \beta)(t) := \alpha(t) \cdot \beta(t)$. One should show that the operation works on homotopy classes but let's just take that for granted.

Observe if $\alpha, \beta, \gamma, \delta$ are loops based at the identity e, then $(\alpha * \beta) \cdot (\gamma * \delta)$ is constructed by going along $\alpha \cdot \gamma$ twice as fast and then concatenation $\beta \cdot \delta$, also going twice as fast. But that is precisely $(\alpha \cdot \gamma) * (\beta \cdot \delta)$, on the nose (we're not even considering homotopy here). By the Eckmann-Hilton argument above, $\pi_1(G)$ is abelian. $\square$

Here is an example:

Example 1.8. Let $f : SU(2) \to G$ be a Lie group morphism. Its kernel is a closed normal subgroup (since f is continuous and the kernel is a preimage of 1) and is either trivial, two elements (the centralizer ± 1), or all of $SU(2)$. I'm using one of the lemmas above to show that nontrivial closed normal subgroups must be in the centralizer.

In the first two cases with finite kernel, then the induced map on Lie algebras is injective as the kernel of df is the Lie subalgebra (and ideal) of $\ker f$. In which case, the pullback of the Killing form of G is a nonzero multiple of the Killing form of $SU(2)$. This is by Cartan's criterion: an ideal of a semisimple Lie algebra is again a semisimple algebra and the Killing form of a semisimple Lie algebra is non-degenerate.

Now, the 3-form $K([x,y],z)$ constructed from the Killing form K generates H^3_{dR} of a simply connected compact simple Lie group G (by Hurewicz, since $\pi_1 = \pi_2 = 0$, $H^3_{dR} \cong \pi_3(G) \otimes \mathbb{R}$).

Hence a Lie group morphism $f : SU(2) \to G$ induces a nullhomotopic map $[f] \in \pi_3(G)$ if and only if it is a trivial Lie group morphism.

2 The Modular Group: $SL(2,\mathbb{Z})$

2.1 The Upper Half Plane

Let H denote the upper half plane in $\mathbb{C}$ without including the real line. Then, we may consider the action of $SL(2,\mathbb{R})$ on H, not by the usual action, but by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \; ad - bc = 1 \qquad \text{then} \qquad A \cdot z = \frac{az+b}{cz+d}.$$

This is a group action as composing Möbius transformations amounts to multiplying matrices in $SL(2,\mathbb{R})$. Of course, A and $-A$ have the same group action in fact, so we could consider $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm I\}$ instead. Also, it is not hard to check that if $\mathrm{Im}(z) > 0$, then $\mathrm{Im}(A \cdot z) > 0$. These transformations are all orientation-preserving as $\det = +1$.

Reducing our attention to $SL(2,\mathbb{Z})$, we claim that this group is generated by two matrices:

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

First, observe that $S^2 = -\mathrm{id}$ while $T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ and $(ST)^3 = \mathrm{id}$. Also, observe that S corresponds to the map $z \mapsto -1/z$ and T corresponds to $z \mapsto z+1$. S is like inversion across the unit circle and then composing with an antipodal map. T is just translation.

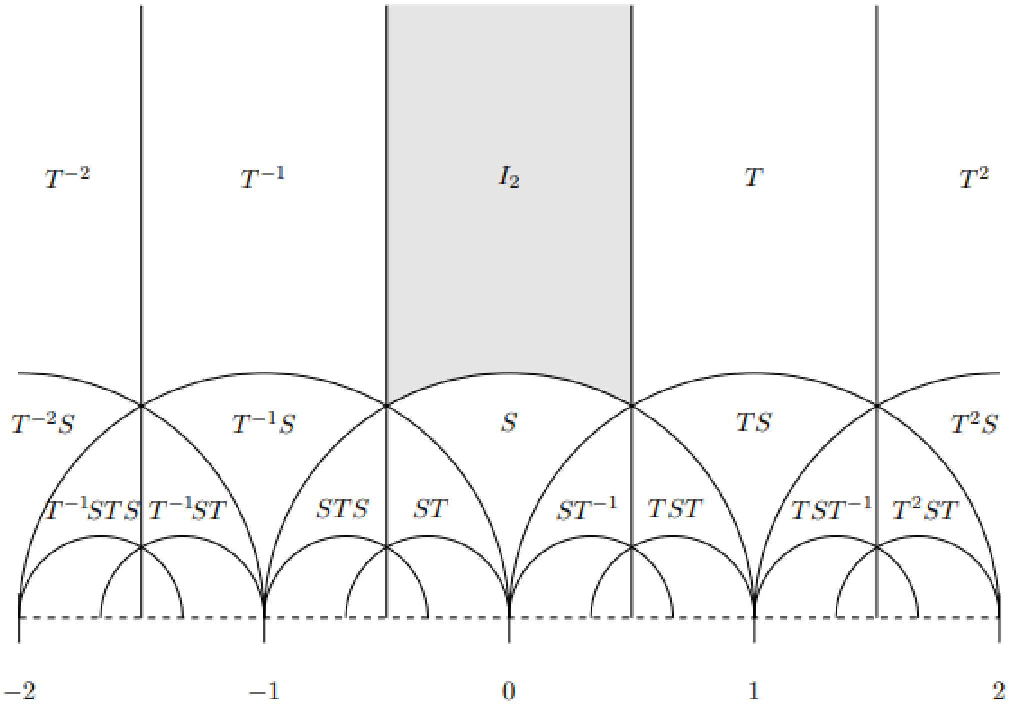
Now, let $G = \langle S, T \rangle$ and $A \in SL(2,\mathbb{Z})$ as above. We show that $G = SL(2,\mathbb{Z})$. Note that

$$SA = \begin{pmatrix} c & d \\ -a & -b \end{pmatrix}, \; T^n A = \begin{pmatrix} a+nc & b+nd \\ c & d \end{pmatrix}.$$

Thus, after multipying by S if needed, we may assume that the entries of A satisfy $|a| \geq |c|$. By the division theorem, $a = qc + r, 0 \leq r < |c|$. Then $T^{-q}A$ has, as its upper left hand entry, $r < |c|$. Multiplying by S interchanges, up to a sign, r and c. Reapplying the division theorem to c and r, we can repeat this process of reducing the upper left hand corner. At the end of the process, we'll have a matrix of the form

$$...ST^{-q_k}T^{-q_{k-1}}...ST^{-q_1}A = \Gamma A = \begin{pmatrix} \pm 1 & m \\ 0 & \pm 1 \end{pmatrix} = T^m, -T^{-m}, \Gamma \in G.$$

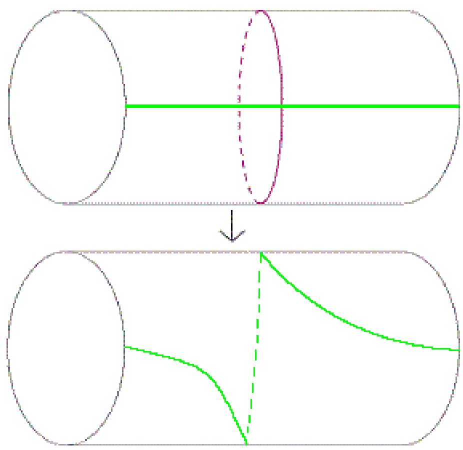
Of course, the diagonal entries have the same sign as $\det = +1$. Since $\Gamma, T^m, -T^{-m} = S^2 T^{-m} \in G$, so then is A. Thus, $G = SL(2,\mathbb{Z})$. it turns out, that if we mod the half plane by $SL(2,\mathbb{Z})$, we get a fundamental domain $\{z \in H : |z| > 1, |\mathrm{Re}(z)| < 1/2\}$. See the figure below; it is from Keith Conrad's notes on $SL(2,\mathbb{Z})$. We can use this domain to tessellate the upper half plane using S, T in a way, reminiscent of hyperbolic fashion. They are of course, linked.



$SL(2, \mathbb{Z})$ acting on the fundamental domain

2.2 The Torus

Let $SL(2, \mathbb{Z})$ act on $H_1(T^2, \mathbb{Z}) \cong \mathbb{Z}^2$ in the usual linear way. Let $\alpha = (1, 0)$ and $\beta = (0, 1)$ be the generators; a latitudinal and longitudinal circle, respectively. Apparently, the terms toroidal and poloidal are also used, respectively. Then S sends $(\alpha, \beta) \mapsto (\beta, -\alpha)$ while T sends $(\alpha, \beta) \mapsto (\alpha + \beta, \beta)$. What T does is takes α and gives it a full rotation along β. One can picture what this does to the torus: cut the torus along β to get a cylinder; twist one end of the cylinder by 2π and then glue it back. Thus, α is made to twist around the torus once; so β is sent back to β but now α has this twist which in homology means $\alpha \mapsto \alpha + \beta$; exactly what we wrote before. This is called a **Dehn twist**. The figure below is from Wikipedia.



A Dehn Twist; the green (horizontal) line is α, the red (vertical) circle is β

As it turns out, $SL(2, \mathbb{Z})$ is the mapping class group of T^2; a mapping class group of a Riemann surface Σ is the group of homeomorphisms modulo continuous deformations and is sometimes called the Teichmüller group. As one suspects, because of Heegard decompositions for 3 manifolds, these groups are important. How does $SL(2, \mathbb{Z})$ act on T^2? We've already said that T twists the torus around α. S can be thought of as puncturing the torus and then turning it inside out (see animation on the Wikipedia page for torus.) But we can also Dehn twist around β and this gives us

$$T_\alpha := T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; T_\beta = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Observe that $T_\alpha^{-1} T_\beta T_\alpha^{-1} = S$. So the mapping class group can be generated from these two Dehn twists.