Basic Properties of K3 Surfaces

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In this note, we review some basic facts about 4-manifolds before considering K3 surfaces over $\mathbb{C}$. These are compact, complex, simply connected surfaces with trivial canonical bundle. It is well known that all K3 surfaces are diffeomorphic to each other by a tough result of Kodaira. The Fermat quartic defined as the zero locus of the polynomial $z_0^4 + z_1^4 + z_2^2 + z_3^2$ in $\mathbb{C}P^3$ can be shown to be simply connected by the Lefschetz Hyperplane theorem because it applies to hypersurfaces as well.

The main resource I used for this note is Mario Micallef and Jon Wolfson's paper *Area Minimizers in a K3 Surface and Holomorphicity.* In the paper, they produce an example of a strictly stable minimal 2-sphere in a non-compact hyperKähler surface that is not holomorphic for any compactible complex structure. This is interesting because, from the Wirtinger inequality, a compact complex submanifold of a Kähler manifold is a volume minimizer in its homology class and any other volume minimizer in the same class must be complex.

1 Topology

Recall that we have a complete topological classification for simply connected, closed, orientable 4-manifolds. The classification depends on the intersection form of such a 4-manifold. Let $Q : A \times A \to \mathbb{Z}$ be a symmetric unimodular bilinear form for some free abelian group A. Unimodular just means that if we have $a \in A$, there exists a $b \in A$ such that $Q(a, b) = +1$.

There is a classification of all such bilinear forms based on their signature, rank, and type. As a reminder, the signature σ is the dimension of the positive eigenspace minus the dimension of the negative eigenspace. The rank is simply the rank of A. Q is of even type if $Q(x, x) \in \mathbb{Z}$ is even for all $x \in A$ and odd, otherwise.

Michael Freedman showed that there is a 1-1 correspondence between closed, oriented, simply connected topological 4-manifolds and symmetric unimodular bilinear forms. The form Q would then be the intersection form of the 4-manifold (we mod out torsion in this discussion). Well, this is almost the theorem. If the type of Q is even, then there is exactly one manifold with Q as intersection form up to homeomorphism. But if Q is odd, there are two non-homeomorphic 4-manifolds with the same intersection form and at least one is not smoothable. To distinguish them, we also need something called a Kirby–Siebenmann invariant. If X, Y are the 4-manifolds with the same odd intersection form, the invariant is most easily defined as being 0 if $X \times S^1$ is smoothable or 1 if not and similarly for Y. X and Y will have different KS invariant.

The upshot of all this is that **topologically, closed, oriented, simply connected 4-manifolds are classified by their intersection forms.** Here are some properties of the intersection form for such 4-manifolds.

1. Suppose that X is a smooth spin 4-manifold; i.e. the 2nd Stiefel-Whitney class vanishes. By Rokhlin's theorem, the signature $\sigma \equiv 0 \pmod{16}$.

2. X is spin if and only if the intersection form Q is even. It is always true that spin 4-manifolds have even intersection form (Wu's theorem) but the converse is false if we do not have the simply connected assumption.

3. The signature is a cobordism invariant for simply connected 4-manifolds. So every 4-manifold is cobordant to some connected sum of $k\mathbb{C}P^2\#\ell\overline{\mathbb{C}P}^2$. This also means, for example, that $\sigma = 0$ if and only if X is the boundary of a compact 5-manifold because, it would be cobordant to S^4.

4. Donaldson: If the intersection form of a smooth simply connected 4-manifold is positive definite, then it is diagonalizable over the integers.

What can we say about K3 surfaces, topologically? The intersection form is:

$$Q = -E_8 \oplus -E_8 \oplus H \oplus H \oplus H.$$

Thus, the intersection form has rank 22, even type, and the dimension of the positive and negative eigenspaces are 3 and 19 respectively. Thus, the signature is -16. K3 surfaces are spin and so the signature is divisible by 16. Quotienting a K3 by an involution gives something called an Enriques surface; these are **not** spin because they have signature divisible by 8 but not 16.

2 Complex Algebraic Structure of K3 Surfaces

Let X be a K3 surface. If we complexify the cohomology, we can extend Q linearly to $\mathbb{C}$. It is well known that we have a Hodge decomposition for compact, complex Kähler manifolds of which K3 surfaces are an example. Thus, $H^2(X, \mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$ where the ranks of these are 1, 20, 1, respectively.

A **marking** of X is a choice of basis of $H^(X, \mathbb{Z})$ or equivalently, a choice of an isometry $\phi : H^2(X, \mathbb{Z}) \to Q$. Here, we're thinking of Q as being a rank 22 free abelian group equipped with the intersection Q in the standard basis. So let's take a marked K3 (X, ϕ). We can complexify $\phi_\mathbb{C} : H^2(X, \mathbb{C}) \to Q_\mathbb{C}$. Since $H^2(X, \mathbb{C})$ decomposes, we can look at just $H^{2,0}(X, \mathbb{C}) = \mathbb{C}\langle\sigma\rangle$. We can choose any generator we like but there is a special $(2, 0)$-form σ called the Bogomolov-Tian-Todorov form which is determined by the marking.

Then $\phi_\mathbb{C}(H^{2,0})$ gives a line in $Q_\mathbb{C}$. Let's consider the subset $\{\Omega \in Q_\mathbb{C} : \Omega \cdot \Omega = 0, \Omega \cdot \overline{\Omega} > 0\}$ of $Q_\mathbb{C}$. σ is in this set and so the complex line $\phi_\mathbb{C}(H^{2,0})$ is in this set. Let's projectivize the set and call it the **period domain** $\mathcal{D}$. Our line $\phi_\mathbb{C}(H^{2,0})$ is now a **point** $[\phi_\mathbb{C}(H^{2,0})]$ in $\mathcal{D}$; call it the **period point** of (X, ϕ).

We'll say that (X, ϕ) and (Y, ψ) are isomorphic as marked K3 surfaces if there exists a biholomorphism $fX \to Y$ which also satisfies $f^*\psi = \phi$. Why is all this interesting? Why introduce markings at all? Here is a major theorem which is called the weak Torelli theorem.

Theorem 2.1. *Two marked K3 surfaces are isomorphic in the sense above if and only if their markings give the same period points.*

Put another way, two unmarked K3 surfaces are biholomorphic if and only if there exists markings on each which give the same period points. One may think of this theorem as saying there is a local isomorphism between the moduli space of marked K3 surfaces and the period domain. The next theorem, perhaps called the strong Torelli theorem, tells us that the period map is surjective:

Theorem 2.2. *All points of the period domain $\mathcal{D}$ occur as period points of marked K3 surfaces.*

For the moment, let's just consider X as a compact, Kähler manifold. A class $\omega \in H^{1,1}(X, \mathbb{R})$ that can be represented as a Kähler form is called a Kähler class. It turns out that Kähler classes satisfy $\omega \cdot \omega > 0$ and $\omega \cdot \sigma = 0$. Note that the set $\{x \in H^{1,1}(X, \mathbb{R}) : x \cdot x > 0\}$ consists of two disjoint connected cones; we call this the **positive cone**.

We let $j : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R})$ be inclusion and the **Picard lattice** is defined to be $S_X = H^{1,1}(X, \mathbb{R}) \cap \operatorname{im} j$. An element $s \in S_X$ is called **divisorial** if there exists a divisor D whose associated line bundle has 1st Chern class s. Additionally, s is called effective if D can be chosen to be effective. Recall, this means that D can be represented as $\sum a_i D_i$ where $a_i > 0$ and D_i are irreducible subvarieties. Also, let a nonsingular curve in X be called nodal if its self-intersection is -2.

The **Kähler cone** is defined to be the convex subcone of the positive cone consisting of those classes that have positive inner product with **any** effective class in S_X. It can be shown that the Kähler cone contains all the Kähler classes.

Theorem 2.3. *For a K3 surface X, the Kähler cone has a simple description. It consists of the classes $\omega \in H^{1,1}(X, \mathbb{R})$ that satisfy: (1) $\omega \cdot \omega > 0$, (2) $\omega \cdot \sigma = 0$, (3) $\omega \cdot \gamma > 0$ for all nodal curves γ in X.*

Because of the surjectivity of the period map, every class in the Kähler cone of a K3 is a Kähler class. Consequently, Yau's theorem can be stated as follows:

Theorem 2.4. *Let (X, ω) be a K3 surface where $\omega \in H^{1,1}(X, \mathbb{R})$ is in the Kähler cone. Then there is a* ***unique*** *hyperKähler metric on X whose Kähler form represents the class ω.*

The adjunction formula tells us that if X is a Kähler surface with a possibly singular holomorphic curve Σ of genus g, then $\Sigma \cdot \Sigma \geq c_1(X) \cdot \Sigma + 2g - 2$. We have equality when Σ is nonsingular. When X is a K3, since $c_1 = 0$, then the inequality becomes $\Sigma \cdot \Sigma \geq 2g - 2 \geq -2$. Thus, if $\Sigma \cdot \Sigma = -2$, then the genus must be $g = 0$ and Σ must be nonsingular.

Since K3 surfaces are hyperKähler and thus, also Calabi-Yau, one may hope to try these techniques with the period map towards understanding other compact hyperKähler manifolds. However, there are more challenges that come about.