Heegaard Floer Theory

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1 Background

The basic idea of a TQFT is that it's a functor from a certain category of cobordisms to the category of vector spaces/modules. Heegaard Floer theory is some package of $(3+1)$-TQFT's that come in four flavors. We begin with a closed, connected, oriented 3-manifold M and we get some invariant $HF^0(M)$. If we had a 4-dim cobordism W between M_1 and M_2, then we should get a map $F_W^0 : HF^0(M_1) \to HF^0(M_2)$. Here, the 0 is a placeholder for the four flavors. We can have $\widehat{HF}, HF^+, HF^-, HF^\infty$ which all amount to different rings for the modules.

1. $\widehat{HF}$ uses the field $\mathbb{F}_2$.
2. HF^- uses the polynomial ring $\mathbb{F}_2[U]$
3. HF^+ uses $T^+ := \mathbb{F}_2[U, U^{-1}]/I$, I is some ideal...
4. HF^∞ uses $\mathbb{F}_2[U, U^{-1}]$

Before giving a definition, historically, we have on 3-manifolds, I_* which is instanton Floer homology and monopole Floer theory which is also called Seiberg-Witten Floer theory (I think). The latter has three flavors which correspond to $HF^\pm$ and HF^∞. Instanton Floer theory is basically about studying the Chern-Simons functional; its critical points are connected by trajectories and the trajectories are in 1-1 correspondence with instantons on $M \times \mathbb{R}$ (from Yang-Mills).

A 3-manifold admits a Heegaard splitting and we can look at flat connections on the surface; the moduli space is a symplectic manifold and the flat connections on the pieces that the surface bounds form Lagrangian submanifolds in the moduli space. So we can do Lagrangian Floer theory on these two Lagrangians. The Atiyah-Floer conjecture says there is an isomorphism between I_* and the Lagrangian Floer theory.

The inspiration for HF^+ comes from thinking about an analogous Atiyah-Floer conjecture for monopole Floer homology.

2 Definition

Now for a definition: Consider a 3-manifold Y with a self-indexing Morse function $f : Y \to [0, 3]$. Then consider $f^{-1}(3/2)$; this is a smooth hypersurface and thus, is some surface Σ_g of genus g. This means that we can break Y into three pieces: $Y^+ = f^{-1}(3/2, 3]$, Σ and $Y^- = f^{-1}[0, 3/2)$. $Y^\pm$ are 3-manifolds with Σ as boundary; so one of them should be homeomorphic to the filled in surface Σ. However, then the complement can be seen to also be a filled in surface Σ. This tells us that Y^+ contains g critical points of index 2 while Y^- contains g critical points of index 1. The unstable manifolds of the index 2 critical points and the stable manifolds of the index 1 critical points are all 2-disks. Thus, they intersect Σ in closed circles. Thus, we have cycles $\alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g$ (let the α_i be from the index 1 critical points).

Now, let take $\Sigma^g = \Sigma \times ... \times \Sigma$, a g-fold product. We can mod this by the symmetric group S_g to obtain a space $\mathrm{Sym}^g(\Sigma)$. Though the action is **not** free, it turns out that this is not such a problem. Moreover, there is a symplectic form on Σ and thus, on the g-fold product; it descends to a symplectic form Ω on $\mathrm{Sym}^g(\Sigma)$. On the other hand, we can consider the tori T_α and T_β where $T_\alpha = \alpha_1 \times ... \times \alpha_g$ and T_β is defined similarly. T_α and T_β are both Lagrangian submanifolds in $\mathrm{Sym}^g(\Sigma)$.

This defines for us Heegaard Floer homology. It is simply doing Lagrangian Floer theory on these tori. $HF^*(Y) := FH^*(T_\alpha, T_\beta)$; the different flavors come from just defining the chains as free modules over the different rings from above.

It turns out that the homology is independent of the Heegaard splitting (so, independent of the Morse function). I'm not sure how to prove this. The idea is that if you have two Heegaard diagrams (which, it seems to me, is just the data from the Morse function), you can show that the chain complex is invariant under Heegaard moves:

- Isotopy
- Handle slides
- Stabilization

But it is also invariant under a non-Heegaard move: diffeomorphisms. This is kind of spectacular: the chains, not just the homology, are invariant.

An interesting question is: suppose we have a sequence of Heegaard diagrams which loops back to the first one; one can move from one diagram to the next via Heegaard moves. Then, we have some kind of a monodromy which induces an automorphism on $HF^0(M)$. The question is; is this automorphism the identity? The answer is yes, proved by Juhász-Thurston (Dylan Thurston).

Note, also: when we count holomorphic disks in Lagrangian Floer theory, we are looking at $u : D \to \mathrm{Sym}^g(\Sigma)$. We have a map from $\mathrm{Sym}^{g-1}(\Sigma) \times \Sigma \to \mathrm{Sym}^g(\Sigma)$ and so u lifts (as D is contractible). There is then projection from $\mathrm{Sym}^{g-1}(\Sigma) \times \Sigma \to \Sigma$; so we get a map $D \to \Sigma$. These maps can be studied combinatorially, at least for the $\widehat{HF}$ flavor.

3 $Spin^c$ Structures

All closed, orientable 3-manifolds are parallelizable: i.e. have trivial tangent bundle. Thom proved that a manifold is a boundary if and only if all its Pontryagin and Stiefel-Whitney classes vanish. So all closed, orientable 3-manifolds are the boundary of some 4-manifold. Also, all compact Lie groups are the boundary of something.

All oriented 4-manifolds admit $Spin^c$ structures so we can just give our 3-manifold a $spin^c$ structure by restricting. But of course, since the 2nd Stiefel-Whitney class is known to vanish for closed, oriented 3-manifolds, then they admit spin and $spin^c$ structure. The point of discussing this restriction from a 4-manifold is that we may consider a 4-dim cobordism W with $spin^c$ structure s between M_1 and M_2 and this should induce some map $F^0_W : HF^0(M_1, s|_{M_1}) \to HF^0(M_1, s|_{M_2})$.

I couldn't really understand what was going on but I guess when you introduce the $spin^c$ structure, Heegaard Floer homology isn't simply singular homology.

4 Results

Kutluhan, Lee, and Taubes (2010) announced a proof that Heegaard Floer homology is isomorphic to Seiberg-Witten Floer homology, and Colin, Ghiggini, and Honda (2011) announced a proof that the plus-version of Heegaard Floer homology (with reverse orientation) is isomorphic to embedded contact homology.

A knot in a 3-manifold induces a filtration on the Heegaard Floer homology groups, and the filtered homotopy type is a powerful knot invariant, called knot Floer homology. It categorifies the Alexander polynomial and detects knot genus. Using grid diagrams for the Heegaard splittings, knot Floer homology was given a combinatorial construction by Manolescu, Ozsváth, and Sarkar (2009).

The Heegaard Floer homology of the double cover of S^3 branched over a knot is related by a spectral sequence to Khovanov homology (Ozsváth and Szabó 2005).

The "hat" version of Heegaard Floer homology was described combinatorially by Sarkar and Wang (2010). The "plus" and "minus" versions of Heegaard Floer homology, and the related Ozsváth-Szabó 4-manifold invariants, can be described combinatorially as well (Manolescu, Ozsváth, and Thurston 2009).