Examples from Differential Geometry

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1 Differential Geometry

Example 1.1. Consider $\mathbb{R}^2$ and the Laplacian $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. If $f(x, y) = \sin(t(x + y))$, then $\Delta f = -2t^2 f$. Thus, the spectrum of Δ (eigenvalues) includes at least $(-\infty, 0]$.

However, consider a smooth, compact, connected Riemannian manifold (M, g), the Laplacian defined as $\Delta = dd^* + d^*d$ where d^* is the formal adjoint of d. It is defined for differential forms and in particular, smooth functions. But for a function, $d^*f = 0$. Thus, $\Delta f = d^*df$. It turns out that the spectrum is non-negative, discrete, and diverge to infinity (so there are infinitely many). This last fact was proved by Hermann Weyl.

Example 1.2. This is the basic example of a closed but not exact form on $\mathbb{R}^n$.

$$\omega = \frac{1}{|\vec{x}|} \sum_i^n x_i(*dx^i)$$

where $*$ is the Hodge-* operator. E.g. $*dx_2 = -dx^1 \wedge dx^3 \wedge dx^4 \wedge ... \wedge dx^n$.

It's not hard to compute that $d\omega = 0$. Alternatively, if we let $f(x_1, ..., x_n) = \frac{1}{2}(x_1^2 + ... + x_n^2)$, then $X := \nabla f$ is the radially outward pointing vector field and $\omega = \iota_X(\frac{1}{|x|} dVol)$. Thus, $d\omega = d\iota_X(\frac{1}{|x|} dVol) = \mathcal{L}_X(\frac{1}{|x|} dVol)$ because $d(dVol) = 0$. I don't know how to see from this that the Lie derivative is zero.

To see that it is not exact, suppose that $\omega = d\eta$ and let

$$\psi = \sum_i^n x_i(*dx^i).$$

Note that ψ is defined on all of $\mathbb{R}^n$ and that $\psi|_{S^{n-1}} = \omega|_{S^{n-1}}$. Also, $d\psi = n\, dVol$ so then

$$nVol(B^n) = \int_{B^n} d\psi = \int_{S^{n-1}} \psi = \int_{S^{n-1}} \omega = \int_{S^{n-1}} d\alpha = \int_{\partial S^{n-1}} \eta = 0.$$

This is a contradiction. Observe that we used Stokes' Theorem twice.

2 Riemannian Geometry

Example 2.1. [Cartan–Hadamard] Let (M, g) be a Riemannian manifold and u, v be two linearly independent tangent vectors at the same point. The sectional curvature is defined as:

$$K(u, v) = \frac{\langle R(u, v)v, u\rangle}{|u|^2|v|^2 - \langle u, v\rangle^2}$$

Here R is the Riemann curvature tensor.

The theorem asserts that the universal covering space of a connected complete Riemannian manifold (M, g) of non-positive sectional curvature is diffeomorphic to $\mathbb{R}^n$. In fact, for complete manifolds on non-positive curvature the exponential map based at any point of the manifold is a covering map.

Example 2.2. [Bonnet-Myers] If Ricci curvature of an n-dimensional complete Riemannian manifold (M, g) is bounded below by $(n-1)k > 0$, then its diameter is at most $\pi/\sqrt{k}$. In particular, this shows that any such M is necessarily compact.

Example 2.3. [Gauss-Bonnet-Chern] Consider a closed, orientable $2n$-dimensional Riemannian manifold (M, g). Let Ω be the curvature form of the Levi-Civita connection of g and $\mathrm{Pf}(\Omega)$, the Pfaffian of Ω.

If we have a skew-symmetric matrix A, its determinant can always be written as the square of an integer polynomial in the matrix entries; the polynomial only depends on the size of the matrix. Then, the Pfaffian is defined as $\mathrm{pf}(A)^2 = \det A$; if A is a $(2n+1) \times (2n+1)$ matrixs, the Pfaffian is zero. This is why we take even dimensions.

We can obviously define the Pfaffian for a 2-form as well since it is skew-symmetric. Now let $\chi(M)$ be the Euler characteristic and define the Euler class by $e(\Omega) = \mathrm{Pf}(\Omega)/(2\pi)^n$. The theorem states that:

$$\chi(M) = \int_M e(\Omega).$$

Observe that in the case of odd-dimensional manifolds, the Euler characteristic is 0 which agrees with the Pfaffian being zero.

Note that this is a generalization of the usual Gauss-Bonnet theorem for surfaces:

$$2 - 2g = \int_{\Sigma_g} K dA$$

where K is the Gaussian curvature defined as $K = \kappa_1 \kappa_2$, where these are the principal curvatures.

3 Seiberg-Witten Theory

Example 3.1. Question: Consider the smooth manifold $M = \#_l \mathbb{C}P^n \#_k \overline{\mathbb{C}P}^n$. $\overline{\mathbb{C}P}^n$ is $\mathbb{C}P^n$ with reverse orientation. When does it admit a symplectic structure? When does it admit an almost complex structure? When does it admit an integrable almost complex structure?

First, for a general smooth 4-manifolds M we can consider the intersection form on $H^2(M;\mathbb{Z})$. We may diagonalize them rationally at least and obtain a signature. Let b_2^+ denote the number of positive diagonal entries. It is a fact in Seiberg-Witten theory that if $X = M\#N$, and both M and N have $b_2^+ \geq 1$, then $SW(X) = 0$. Also, b_2^+ is additive; meaning, $b_2^+(X) = b_2^+(M) + b_2^+(N)$. Lastly, $b_2^+(\mathbb{C}P^2) = 1$ as $H^2(\mathbb{C}P^n;\mathbb{Z}) = \mathbb{Z}$; in Dobeault cohomology, it is generated by the Fubini-Study form ω. Let us now turn to our case here.

For $l = 1$, these are blowups of the projective plane, which are all Kähler and hence symplectic. For $l > 1$, these do not have symplectic structures. For if l is even, then they don't even have almost complex structures (I don't understand the reasons behind this), but symplectic manifolds certainly are all almost complex. If $l > 1$ is odd, then their Seiberg-Witten invariants would vanish, since your manifolds decompose as a connected sum into pieces with positive b_2^+. For example, if $l = 3, k = 0$, then $X = \mathbb{C}P^2\#\mathbb{C}P^2\#\mathbb{C}P^2$ has $b_2^+ = 3$ and $SW(X) = 0$.

But a famous theorem of Taubes says that symplectic 4-manifolds have non-vanishing Seiberg-Witten invariants. Finally, if $l = 0$, then $b_2^+ = 0$, so your manifold cannot be symplectic, since the cohomology class of the symplectic form has positive square.