Examples Concerning Bundles

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1 Vector Bundles

Example 1.1. How do we know whether TS^{2n} is trivial? Suppose it is: $TS^{2n} \cong S^{2n} \times \mathbb{R}^{2n}$. This would mean we could smoothly choose a nonzero vector at each point, say, $(1, 0, ..., 0) \in \mathbb{R}^{2n}$. But then we have a smooth non-vanishing vector field at each point of S^{2n} which contradicts the Poincaré-Hopf theorem (specifically, the Hairy Ball Theorem).

1.1 Characteristic Classes

Example 1.2. For a fixed rank k, there are only two vector bundles for S^1. An orientable one which is just the trivial bundle $S^1 \times \mathbb{R}^k$ and an non-orientable one which is the Möbius bundle μ direct sum with a trivial bundle: $\mu \oplus \mathbb{R}^{k-1}$. Then, for a manifold M, we may then consider a map $w_1 : \pi_1(M) \to \mathbb{Z}_2$. Take an element $[f] \in \pi_1(M)$; it is a homotopy class of $f : S^1 \to M$. Then the image of f_* is a subbundle of TM. We can then consider the normal bundle ν_f. If it is trivial, $w_1([f]) = 0$, if non-orientable, then $w_1([f]) = 1$.

Claim: $w_1([f * g]) = w_1([f]) + w_1([g])$. In particular, we see that $[f * f] \mapsto 0$. Thus, $w_1 \in \mathrm{Hom}(\pi_1(M), \mathbb{Z}_2) \cong H^1(M, \mathbb{Z}_2)$. This is, in fact, the 1st Stiefel-Whiteney class.

1.2 Constructions

Fiber bundles for spheres can be constructed explicitly using the **Clutching construction.** Let G be a Lie group and $b : S^{m-1} \to G$ represent an element of $\pi_{m-1}(G)$. Since S^m is given by two balls glued together along their boundary, we have open subsets $U_1, U_2 \subset S^m$ diffeomorphic to open m-balls so that $U_1 \cap U_2$ deformation retracts on to $S^{m-1} \subset S^m$. Hence we have a map $\Phi_{12} : U_1 \cap U_2 \to G$ whose restriction to S^{m-1} is b. This gives the transition data for the fiber bundle over S^m representing $[b] \in \pi_{m-1}(G)$.

Example 1.3. Let us show that all rank 3 $\mathbb{R}$-vector bundles over S^3 are trivial. In this case the structure group is $GL(3)$ which is homotopic to $O(3)$. We can calculate $\pi_2(O(3))$ as follows: The connected component of $O(3)$ containing id is $SO(3)$ so it is sufficient to compute $\pi_2(SO(3))$. Each element of $SO(3)$ is a rotation about some axis.

This means that we have a fibration

$$\begin{tikzcd} SO(2) - \{\mathrm{id}\} \arrow[r, hook] & SO(3) - \{\mathrm{id}\} \arrow[d, "q"] \\ & \mathbb{R}P^2 \end{tikzcd}$$

q sends $\rho \in SO(3) - \{\mathrm{id}\}$ to the corresponding unique axis of rotation of ρ. The fiber is $SO(2) - \{\mathrm{id}\} \cong S^1 - \{1\} \cong (0, 1)$ because it stabilizes the axis of rotation. So we have the following homotopy LES:

$$\ldots \to \pi_k((0,1)) = 0 \to \pi_k(SO(3) - \{\text{id}\}) \to \pi_k(\mathbb{R}P^2) \to 0 = \pi_{k-1}((0,1)) \to \ldots$$

which implies that $\pi_k(SO(3) - \{\text{id}\}) \cong \pi_k(\mathbb{R}P^2)$.

Now $\pi_2(\mathbb{R}P^2) = \pi_2(S^2) = \mathbb{Z}$ (a covering map induces isomorphisms on higher homotopy groups) and hence $\pi_2(SO(3) - \{\text{id}\}) = \mathbb{Z}$. The generator of $\pi_2(\mathbb{R}P^2)$ is the covering map (antipodal map) $\alpha : S^2 \to \mathbb{R}P^2$ and hence the generator of $\pi_2(SO(3) - \{\text{id}\})$ is a lift $\tilde{\alpha} : S^2 \to SO(3) - \text{id}$ of α to $SO(3) - \{\text{id}\}$. Note that we may lift since S^2 is simply connected.

$$\begin{tikzcd} & SO(3) - \{\text{id}\} \arrow[d, "q"] \\ S^2 \arrow[ur, "\tilde{\alpha}"] \arrow[r, "\alpha"] & \mathbb{R}P^2 \end{tikzcd}$$

Here $\tilde{\alpha}(v)$ is defined as a clockwise rotation of angle $\theta \in (0, 2\pi)$ about the axis v. It doesn't matter so much what θ is; as long as $\theta \neq 0, 2\pi$, all the maps are homotopic. If we choose θ to be small then the image of $\tilde{\alpha}$ is a small sphere near the point id $\in SO(3)$. This means that $\tilde{\alpha}(S^2)$ is contained in a small chart of $SO(3)$ and hence is contractible.

Hence the map $\iota_* : \pi_2(SO(3) - \{\text{id}\}) \to \pi_2(SO(3))$, induced by inclusion, is 0. Also since $SO(3)$ is 3-manifold, the map ι_* is surjective as any map $h : S^2 \to SO(3)$ can be perturbed to a map $\hat{h} : S^2 \to SO(3) - \{\text{id}\}$. Hence $\pi_2(SO(3)) = 0$. Therefore, every rank 3 $\mathbb{R}$-vector bundle on S^3 is trivial as told by the Clutching construction.

Example 1.4. While we're on the subject of $SO(3)$, let us consider the map $f : \mathbb{C}^3 \to \mathbb{C}$ given by $f(x, y, z) = x^2 + y^2 + z^2$. Then $A := f^{-1}(0)$ is a singular hypersurface of $\mathbb{C}^3$ with singularity at 0. If we take a point λ near 0, then it turns out that $f^{-1}(\lambda)$ is symplectomorphic to T^*S^2. As $\lambda \to 0$, the zero section, which is just the cycle S^2, vanishes. But away from the zero section, the fibers still behave well.

All this to say, if we take the unit sphere bundle of T^*S^2 this is the same as the link L_A of the hypersurface A which by definition, is $A \cap S_\epsilon$; some small ϵ-sphere. Let's show that this link L_A is diffeomorphic to $\mathbb{R}P^3$ which is of course, diffeomorphic to $SO(3)$. To establish the latter claim, just see that $Spin(3) = S^3$ is the double cover of $SO(3)$ but also the double cover of $\mathbb{R}P^3$.

To show that $L_A \cong S(T^*S^2) \cong SO(3)$, recall that the elements of $SO(3)$ are triples $(v_1, v_2, v_3) \in (\mathbb{R}^3)^3$ which form oriented orthonormal bases. We will treat T^*S^2 as TS^2 since we're only looking at the smooth topology of the total spaces and not the full vector bundle data. Thus, take a point $v_1 \in S^2 \subset \mathbb{R}^3$ and a point $x \in T_{v_1}S^2$. This a vector tangent at the point v_1. The pair $(v_1, x) \in S(T^*S^2)$; we will show that this maps to a point of $SO(3)$ via a construction. When we include x into $\mathbb{R}^3$, we obtain a vector v_2 which is orthogonal to v_1. Lastly, let $v_3 = v_1 \times v_2$. Since we care about orientation and also require the norm of all three vectors to be 1, v_3 is completely determined by v_1 and v_2. Therefore, the map $(v_1, x) \mapsto (v_1, v_2, v_3)$ is a diffeomorphism.

2 Leray-Hirsch Theorem

Recall that on a smooth manifold M, a **good cover** is a collection of open sets $\{U_a\}$ such that any finite intersection $U_{a_1} \cap \ldots \cap U_{a_n}$ is homeomorphic to $\mathbb{R}^n$. All manifolds admit good covers; the proof given in Bott-Tu uses some Riemannian geometry and geodesically convex sets.

A manifold is of finite type if it admits a finite good cover. For example, if the manifold is compact, it is of finite type.

Theorem: Let $\pi : E \to M$ be a fiber bundle over M with fiber F. Suppose M has a finite good cover. Suppose there are global cohomology classes $e_1, ..., e_r$ on E which restrict to a basis of the cohomology of each fiber; i.e. $\iota^* : H^*(E) \to H^*(F)$ is surjective. Then we can define a map $\psi : H^*(M) \otimes H^*(F) \to H^*(E)$ which is an isomorphism of $H^*(M)$-modules.

So this theorem tells us that $H^*(E)$ is a free module over $H^*(M)$. The theorem actually works for general fiber bundles with singular cohomology over $\mathbb{Q}$. If ι^* is a surjection, then let $s : H^*(F) \to H^*(E)$ be a section; i.e. $\iota^* \circ s = \mathrm{id}$. The map $\psi : H^*(M) \otimes H^*(F) \to H^*(E)$ can be defined as $(\alpha, \beta) \mapsto \pi^*\alpha \cup s(\beta)$.

3 Principal G Bundles and Connections

Example 3.1. There are topological obstructions to a vector bundle admitting a flat connection: most simply, by Chern-Weil theory the real Pontryagin classes of such a bundle must all vanish. So, for example, any closed 4-manifold with nonzero signature, such as $\mathbb{C}P^2$, does not admit a flat connection. Recall that the signature of a 4-manifold is the number of positive eigenvalues minus the number of negative eigenvalues of the intersection form.

Example 3.2. By Chern-Weil theory, or by the Chern-Gauss-Bonnet theorem (which is stated on Wikipedia for the Levi-Civita connection but in fact holds for any connection), if an oriented vector bundle admits a flat connection, then the real Euler class must also vanish, meaning that the Euler characteristic must be zero. So it follows that the tangent bundle of S^2 does not admit a flat connection as $\chi(S^2) = 2$; cf. hairy ball theorem.

Example 3.3. [Hopf Fibration] This is a very important example. As you know, it was the first construction which shows that $\pi_n(S^k)$ can be nontrivially, even if $n > k$.

Let's construct the Hopf fibration by embedding $S^3 \subset \mathbb{C}^2$; then S^1 acts naturally on $\mathbb{C}^2$ by the diagonal action. In fact, if we cut S^3 by an vector space embedding of $\mathbb{C} \subset \mathbb{C}^2$, we see that the intersection of $\mathbb{C} \cap S^3$ is a copy of S^1. For example, if we see where $S^3 \cap \{(z, 0) : z \in \mathbb{C}\}$, then the defining equation $|z|^2 + |w|^2 = 1$ is now $|z|^2 = 1$. With this construction, we find that in fact, S^3 is the restriction of the tautological line bundle over $\mathbb{C}P^1$ to unit length vectors. This means $S^3 = D\xi$ or sometimes it may be written as $D\mathcal{O}(-1)$. So the associated line bundle to this principal S^1-bundle is the tautological line bundle over $\mathbb{C}P^1$.

There's also an explicit map. If we identify $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$, then let $p(z, w) = (2z\bar{w}, |z|^2 - |w|^2)$.

$$\begin{tikzcd} S^1 \arrow[r, hook] & S^3 \arrow[d, "p"] \\ & \mathbb{C}P^1 \end{tikzcd}$$

Another interesting property of this Hopf fibration is that, if we think of S^3 as the one-point compactification of $\mathbb{R}^3$, then we can fill $\mathbb{R}^3$ with disjoint circles and a single line. That is, every point in $\mathbb{R}^3$ lies on a circle or line. This isn't that hard to do under these conditions. But now add a further condition: the circles should all be pairwise linked. This is less trivial but the Hopf fibration gives this to us.

A flat connection ∇ on a principal fiber bundle $P \to M$ with structure group G defines a homomorphism from the fundamental group of M to G via parallel transport along closed curves. (Otherwise put, the holonomy group of a flat connection is a homomorphic image of the fundamental group.) This defines a reduction of the structure group G to this so called monodromy group. In particular, if M is simply connected, then the monodromy group is trivial which means the bundle is trivial.

The Hopf fibration does not admit a flat connection. This is because: if it admits a flat connection and and the base is simply connected, then the bundle is trivial. But the bundle is not trivial: the pullback of the standard volume form on S^2 is $d\theta$ where θ is the standard contact from on S^3. Moreover, $d\theta$ is also the Euler class. Since the Euler class is nonzero, the bundle is not trivial. We conclude that the Hopf fibration does not admit any flat connections. (The nontriviality of the bundle can also be seen from Chern-Weil theory, which computes characteristic classes from curvature expressions).

Perhaps another intuitive way to see this is that S^2 is curved and by Gauss-Bonnet, we must have positive curvature somewhere on S^2. Now, this is of course with respect to curvature of connections on vector bundles, not S^1-bundles. But I think there is an isomorphism between $\mathbb{C}$-line bundles (which we might think of as rank 2 real vector bundles) over S^2 and S^1 bundles over S^2. The former is classified by $H^2(S^2) = \mathbb{Z}$, the latter by $\pi_1(S^1) = \mathbb{Z}$.

By the way, using the homotopy long exact sequence, we can use the Hopf map to induce an isomorphism $\mathbb{Z} \cong \pi_3(S^3) \cong \pi_3(S^2)$. The generator of $\pi_3(S^3)$ is simply the identity map $\mathrm{id} : S^3 \to S^3$ and this is mapped to the generator of $\pi_3(S^2)$; so the generator is the homotopy class of the Hopf map.

One last note; of course the Hopf fibrations work for higher dimensional odd spheres:

$$\begin{tikzcd} S^1 \arrow[r, hook] & S^{2n+1} \arrow[d, "p"] \\ & \mathbb{C}P^n \end{tikzcd}$$

Note that if we have the antipodal map $\alpha : \mathbb{C}^2 \to \mathbb{C}^2$, $(z, w) \mapsto (-z, -w)$, we can quotient S^1 and S^3 by α to obtain a $\mathbb{R}P^3$ as a $\mathbb{R}P^1$-bundle over S^2. But $\mathbb{R}P^1 = S^1$ so $\mathbb{R}P^3$ can be viewed as a principal S^1-bundle over S^2.

Example 3.4. It is well known too that the Hopf fibrations extend to looking at $\mathbb{H}$ and $\mathbb{O}$. For example, we have

$$\begin{tikzcd} S^3 \arrow[r, hook] & S^7 \arrow[d, "p"] \\ & S^4 \end{tikzcd} \qquad \begin{tikzcd} S^3 \arrow[r, hook] & S^{4n+3} \arrow[d, "p"] \\ & \mathbb{H}P^n \end{tikzcd}$$

However, with the octonions, we only get one fibration due to the non-associativity of the octonions.

$$\begin{tikzcd} S^7 \arrow[r, hook] & S^{15} \arrow[d, "p"] \\ & S^8 \end{tikzcd}$$

Somewhat related, we have a symplectic fibration over a non-symplectic base:

$$\begin{tikzcd} S^2 \arrow[r, hook] & \mathbb{C}P^3 \arrow[d, "p"] \\ & S^4 \end{tikzcd}$$

Example 3.5. If we have a line bundle, then a flat connection ∇ can be written locally as $d + \alpha$, α is a 1-form. ∇ has a nowhere vanishing parallel section if and only if α is exact. As a result, there can also be flat connections with no nontrivial parallel sections. For example, take $\nabla = d + \alpha$ on the trivial bundle of S^1, where α is the generator of $H^1(S^1)$. It does not have a parallel section. By the way, sometimes α is misleadingly written as $d\theta$ which makes it appear to be exact but of course, there is no global coordinate function θ on S^1.