HMS $D^b_{coh}$ is triangulated so we need to

# Twisted Complexes & Split Generation

Recall from Ben's talk on the derived category: a triangle is some triple of objects w/ morphisms

$A \xrightarrow{f} B \to C \to A[1]$ & is *distinguished* if $C \underset{q.i.}{\simeq} C(f)^i = \left( A^{i+1} \oplus B^i, \begin{pmatrix} d_A & 0 \\ f & d_B \end{pmatrix} \right)$.

morphism induces isomorphism on the cohomology level.

Analogously in an $A_\infty$-cat $\mathcal{A}$

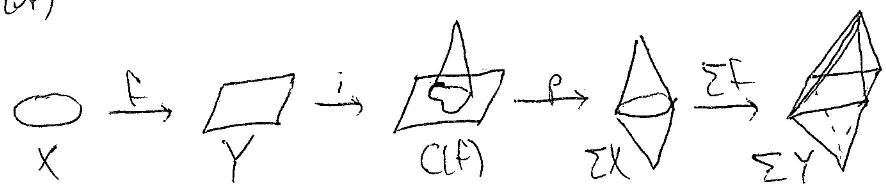
def: An *exact triangle* is a triple of objects $A, B, C$ & closed morphisms $f, g, h$ s.t. $C \underset{q.i.}{\simeq} C(f)$ & $g$ & $h$ are natural maps to & from it.

$\mu^1(f) = 0$

Exactness means each composition in coh cat $H(\mathcal{A})$ are zero: $\mu^2(g,f)$, $\mu^2(h,g)$, $\mu^2(f,h)$.

As in $D^b_{coh}$: exact triangles give LES in $H(\mathcal{A})$

Model Example: let $f: X \to Y$ be a continuous map on topological spaces:




Each composition is nullhomotopic; i.e. in cohomology, we have a LES.

This is a model example b/c for chain cplxs & $f: A \to B$ is a chain map, $A[1] \leftrightarrow \Sigma A$, $C(f) = A[1] \oplus B$

$i$ includes $B \hookrightarrow C(f)$, $p$ projects $C(f) \to A[1]$

---

def: An $A_\infty$-cat is triangulated if every closed morphism $f: A \to B$ can be completed to an exact triangle & the shift functor is a quasi-equivalence (it acts on gradings); all morphisms have mapping cones.

Compare triangulated Cat w/ $A_\infty$-triangulated Cat:

- triangles are additional structure
- the $A_\infty$-structure is rich enough to know about triangles & not treat them as additional structure
- $A_\infty$ functors always map exact triangles to exact triangles

Remember: HMS is about $D^b_{coh}$, on the B side, which is triangulated. So we need to triangulate $\mathcal{F}(M)$ somehow.

# Twisted Complexes (write def before the talk, to the side)

A twisted cplx gives us a way to obtain a triangulated $A_\infty$-cat from an $A_\infty$-cat $\mathcal{A}$.

Moreover, $\mathcal{A} \hookrightarrow Tw\mathcal{A}$ fully faithfully.*

(refer to definition): • This strictly lower triangular differential condition really means $\delta = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ \Delta & & 0 \end{pmatrix}$

$\mu^k_{Tw}(\ldots) = \sum\limits_{i_k,\ldots,i_0 \geq 0} \mu^{k+i_0+\cdots+i_k}(\ldots)$ • the twisted operations require the original higher operations; the sum is finite b/c of the strictly lower triangular condition.

e.g. Say we have $A \overset{f}{\curvearrowright} B \overset{g}{\curvearrowright} C$. Consider $(A[2] \oplus B[1] \oplus C, \delta = f + g)$. This is a twisted cplx iff

$$\delta = f+g = \begin{pmatrix} 0 & 0 & 0 \\ f & 0 & 0 \\ 0 & g & 0 \end{pmatrix}$$

$\mu^1(f) = \mu^1(g) = 0$ & $\mu^2(g,f) = 0$. (exact on the nose)

we can instead introduce $h \in hom^{-1}(A,C)$: $\delta = \begin{pmatrix} 0 & 0 & 0 \\ f & 0 & 0 \\ h & g & 0 \end{pmatrix}$ & the last condition becomes $\mu^2(g,f) + \mu^1(h) = 0$

$h$: "chain homotopy"

in our setting, $\mu^1(h) = dh + hd$.

So this says $\mu^2(g,f) = 0$ in $H(\mathcal{A})$.

exact in cohomology

**Def:** Given ~~twisted cplxs~~ $(E,\delta), (E',\delta') \in Tw\mathcal{A}$ & closed morph $f \in hom^0(E,E')$ (so $\mu^1_{Tw} f = 0$),

the <u>abstract</u> mapping cone of $f$ is $Cone(f) = \left(E[1] \oplus E', \begin{pmatrix} \delta & 0 \\ f & \delta' \end{pmatrix}\right)$.

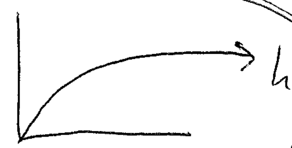
Given obj $A, B, C$ of $\mathcal{A}$ & closed morph $f \in hom^0(A,B)$ ($\mu^1 f = 0$), $C$ is a <u>mapping cone</u> of $f$, if, in $Tw\mathcal{A}$, $C \underset{q.i.}{\cong}$ ~~abstract~~ $Cone(f)$ in Tw (recall the embedding of $\mathcal{A} \hookrightarrow Tw\mathcal{A}$).

# Exact Triangles in the Fukaya Category

Q: What's the relevance of mapping cones to the Fukaya Cat? } HMS

A: Some mapping cones in $\mathcal{F}(M)$ can be understood geometrically. ✓ Main source: Lagrangian Surgery; Dehn twists represent an important case.

Generalized Dehn Twists: Consider $(T^*S^n, \omega = d\lambda)$ & $h: [0,\infty) \to \mathbb{R}$ be s.t. $h'(0) = \pi$, $h'' \leq 0$ & is const outside a nbhd of $0$.



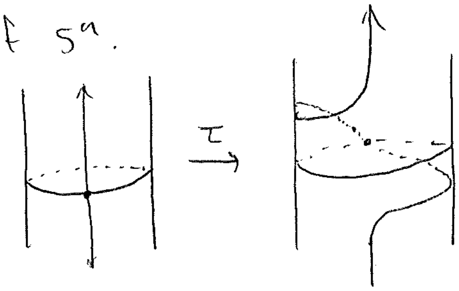

maybe don't say

Let $(q,p)$ be coord & $H: T^*S^n \to \mathbb{R}$, $H(q,p) = h(\|p\|)$ std metric

fiber coord

This gives a symplectomorphism which is the antipodal map on zero section $S^n$ & identity map outside some nbhd of $S^n$.

e.g. $T^*S^1$




$$T^*S^n \cup V \supset S^n$$

Thm (Weinstein): Let $S$ be a Lagrangian sphere in $(M,\omega)$. $\exists$ nbhd $S \subset U \underset{symp}{\cong} V \supset S^n$.

So we can consider Dehn twists of Lagrangian spheres in $M$, not just $T^*S^n$

Thm (Seidel): Given a Lagrangian sphere $S$ & any object $L$, $\exists$ exact triangle in $Tw\,\mathcal{F}(M)$

$$\begin{tikzcd} HF^*(S,L) \otimes S \arrow[r, "ev"] & L \arrow[d] \\ & \tau_S(L) \arrow[ul, "{[1]}"] \end{tikzcd}$$

So $\tau_S(L) \underset{q.i.}{\simeq} \mathrm{Cone}(ev)$.

~~Here $HF^*(S,L) \otimes S = \bigoplus_{g \in HF^*(S,L) \text{ generator}} S_g$ ; this is graded according grading of $HF^*$ ; $ev: S_g \mapsto g \otimes L$~~

~~Note:~~ & we get a LES from this, per usual.

What's ev? For me, I think of it w/ Yoneda's lemma. Let $K$ be a test obj & take hom: $HF^*(S,L) \otimes HF^*(S,K) \xrightarrow{ev} HF^*(L,K)$

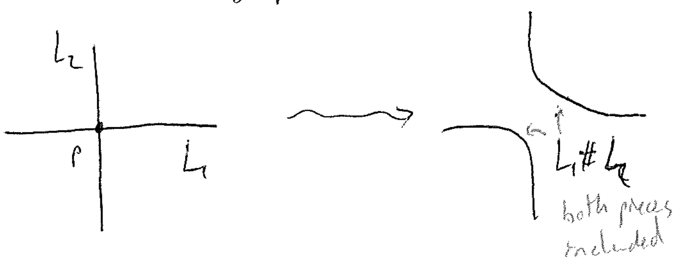
maybe don't say:

# Lagrangian Surgery in General

Given $L_1, L_2$ - Lagrangians w/ $L_1 \cap L_2 = \{p\}$, $\exists$ "Darboux chart" $U \cong (\mathbb{C}^n, \omega_0)$ centered at $p$ w/ $T_pL_1 = \mathbb{R}^n$, $T_pL_2 = (i\mathbb{R})^n$.

Take $\varepsilon > 0$ & define 1-form $\eta = \varepsilon\, d\log\|x\|$ on $\mathbb{R}^n \setminus \{0\}$. The graph of $\eta$ has coord given by $y_i = \frac{\varepsilon x_i}{\|x\|^2}$. maybe skip

General Fact: The graph of a closed 1-form is a Lagrangian. Asymptotically, $\Gamma(\eta)$ approaches the zero section & the cotangent fiber.

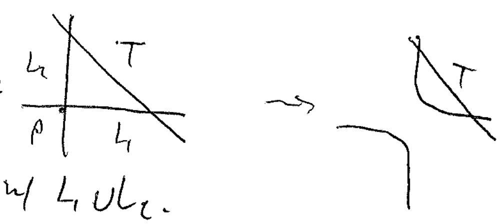



Use cutoff $f^{ns}$ to construct $L_1\#L_2$ which agrees w/ $L_1 \cup L_2$ outside a nghd of $p$. This is, up to Hamiltonian isotopy, independent of $\varepsilon > 0$ & other choices. Not so w/ multiple intersections, however.

Rmk: If $L_2$ = Sphere, $L_1\#L_2 \underset{H.I}{\simeq} \tau_{L_2}(L_1)$.

---

Let $T$ be another Lagrangian generically positioned wrt $L_1$ & $L_2$:




For small $\varepsilon > 0$, the surgery doesn't affect the intersections of $T$ w/ $L_1 \cup L_2$.

The 1-dim picture suggests the following, which is proved in FOOO:

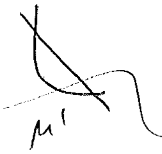
Thm) For suitable $J$ & $\varepsilon > 0$, $J$-hol <u>strips</u> w/ boundary on $T$ & $L_1\#L_2$ connecting an intersection in $T \cap L_2$ to $T \cap L_1$ are in bijection w/ $J$-hol triangles w/ boundary on $T, L_1, L_2$, & a corner on $p$. Counts of rigid strips in the other direction is $0$.

Rmk: this is related to the algebra: the chain maps go in one direction only.

Claim: $CF(T, L_1\#L_2)$ is the mapping cone of $\mu^2(p,\cdot): CF(T, L_2) \to CF(T, L_1)$. So we have an exact triangle

$$\begin{tikzcd} L_2 \arrow[r, "p"] & L_1 \arrow[d] \\ & L_1\#L_2 \underset{q.i}{\simeq} \mathrm{Cone}(p) \arrow[ul, "{[1]}"] \end{tikzcd}$$

Recall: $\mu^1_{Tw}$ required $\mu^1$ & $\mu^2$; this is consistent w/ the description here:




We can study the Fukaya Category & $\mathrm{Tw}\,\mathcal{F}$ via "simple" Lagrangians which give some mapping cones. But then, in $\mathrm{Tw}\,\mathcal{F}$ (as in $\mathcal{F}$), we really need to understand the higher operations.

i.e. we can replace Lag by quasi-iso twisted cplxs built from simpler Lag at the expense of considering higher operations on their Floer cplxs.

---

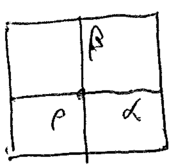
## Generation & Yoneda Embedding

def: Objects $G_1, \dots, G_r$ generate $A_\infty$-cat $\mathcal{A}$ if in $\mathrm{Tw}\,\mathcal{A}$, every obj of $\mathcal{A}$ is quasi-iso to a twisted cplx built from copies of $G_1, \dots, G_r$. i.e. built from $G_1, \dots, G_r$ via iterated mapping cones.

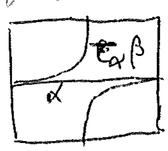
They split generate if every obj is q.i. to a direct sum of twisted cplxs built from copies of $G_1, \dots, G_r$.

---

e.g. $(T^2, \omega)$




Lagrangians here are (embedded) (smooth) closed curves (or unions of them) w/o crossings.




Iterating mapping cones using $\alpha$ & $\beta$ give simple closed curves representing all nontrivial primitive elements in $\pi_1(T^2) = \mathbb{Z}^2$.

i.e. $x = y^n$, $n \geq 2$ is not primitive

e.g. $\tau_\alpha \beta \underset{q.i.}{\simeq} \mathrm{Cone}(p)$.

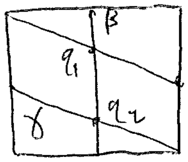
However, these objects all satisfy a balancing condition: Given $\theta \in \Omega^1(T^2 \setminus \{pt\})$ w/ $d\theta = \omega$ & $\int_\alpha \theta = \int_\beta \theta = 0$, $\alpha, \beta$ generate $L$'s which always give $\int_L \theta = 0$.

So $\alpha$ & $\beta$ only generate a subcat of $\mathcal{F}(T^2)$ b/c in each homology class, there is exactly one Hamiltonian isotopy class w/ ~~each~~ $\int \theta$ taking each value of $\mathbb{R}/\mathbb{Z}$. $\alpha, \beta$ generate only $0 \in \mathbb{R}/\mathbb{Z}$.

---

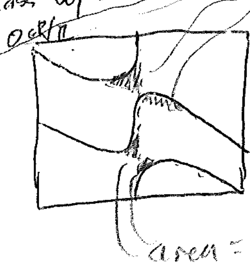
Go to next page

Consider, instead




~~The cone of~~

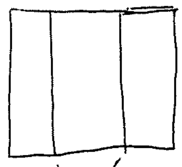
Claim: $\mathrm{Cone}\left(T^{a_1}_{q_1} + T^{a_2}_{q_2}\right) \underset{q.i.}{\simeq}$




The two components are both homotopic to $\alpha$. But by tuning $a_1, a_2$ we can get any value of $\int_\gamma \theta \in \mathbb{R}/\mathbb{Z}$.

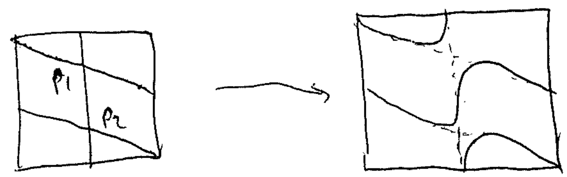
The total curve is balanced but each individually is not; split generation let's us split the total curve into individual curves.

$T^2$

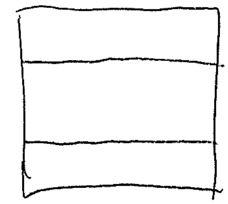


not Ham isotopic.

So in a homotopy class, we have lots of different Ham isotopy classes, $\mathbb{R}/\mathbb{Z}$ of them.
How do we obtain these? Through split generation:



the two components are homotopic to $\alpha$



$l_1$

$l_2$

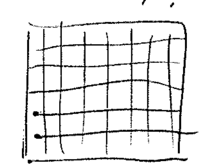
$$L \underset{q.i.}{\simeq} \mathrm{Cone}(T^{a_1}p_1 + T^{a_2}p_2)$$

Split generation let's us split $L$ into two in order to get $l_1$ & $l_2$, individually.

Sort of abstractly: Say $p \in \hom(L, L) = \hom(l_1, l_1) \oplus \hom(l_2, l_2)$ & $p^2 = p$ – idempotent thing.
idempotents in linear algebra, $p: V \to V$ $V$ splits ~~space~~ into $\ker p \oplus \mathrm{Im}\, p$ & similarly, if we add in idempotents, we can obtain $\hom(l_1, l_1)$ & $\hom(l_2, l_2)$ separately.

So we obtain $l_1$ & $l_2$. By tuning $a_1$ & $a_2$, the surgery parameters, we get $\int_{l_1} \theta = x$ for any $x \in \mathbb{R}/\mathbb{Z}$. This gives all the Ham iso classes.

So if we take all of the vertical lines & horizontal lines on the square, they generate $\mathcal{F}(T^2)$.
But $\alpha, \beta$ are enough to split-generate $\mathcal{F}(T^2)$.



Now w/ split generation, we get all the Hamiltonian isotopy classes of curves & that's the full Fukaya cat. So $\alpha$ & $\beta$ split generate $\mathcal{F}(T^2)$.

Yoneda Embedding:

Let $G_1, \dots, G_r$ split-generate $\mathcal{A}$, let $\mathcal{G} = \bigoplus_{i,j=1}^{r} \hom(G_i, G_j)$ be the endomorphism algebra ~~of~~ which is a $A_\infty$-algebra

Structure maps come from $\mathcal{A}$.

Given $L \in Ob_j(\mathcal{A})$, $\mathcal{Y}(L) = \bigoplus_{i=1}^{r} \hom(G_i, L)$ is an $A_\infty$-module over $\mathcal{G}$ w/ some operations.

Given $a \in \hom(L, L')$, $\mathcal{Y}(a) \in \hom_{mod\text{-}\mathcal{G}}(\mathcal{Y}(L), \mathcal{Y}(L'))$ whose linear term is given by composition w/ $a$.

This $\mathcal{Y}$ is an $A_\infty$-functor & is the restriction to $G_1, \dots, G_r$ of the $A_\infty$-Yoneda embedding

$$\mathcal{A} \longrightarrow mod\text{-}\mathcal{A}.$$

Prop: This construction above extends to an $A_\infty$-functor $\mathcal{Y}$ from $\mathcal{A}$ to $mod\text{-}\mathcal{G}$.

If $G_1, \dots, G_r$ split generate $\mathcal{A}$, then $\mathcal{Y}$ is a fully faithful quasi-embedding

I sort of think about this like in linear algebra. You have a basis (these split-generators); understanding how $\mathcal{Y}$ behaves on them tells you about $\mathcal{A}$ but in the context of the easier-to-category $mod\text{-}\mathcal{G}$ which is actually just a differential graded category.