Homological Mirror Symmetry of the Cylinder

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These are notes I took from a talk by Mohamed El Alami for the RTG Student Seminar, Fall 2019.

In the context of mirror symmetry, we may want to enrich the Fukaya category such that the objects are Lagrangians with **local systems**. To a Lagrangian submanifold L, we equip a complex line bundle $E \to L$ with a flat $U(1)$ connection. I'm not sure why we don't consider $U(n)$ in general and only need line bundles. Also, we may as well take the trivial bundle. Then, we need to define Floer theory. The generators of the Floer chains will still be intersection points but now, the differential will be defined differently.

We have that

$$\hom((L_0, \nabla_0), (L_1, \nabla_1)) = \sum_{p \in L_0 \cap L_1} \hom(\mathbb{C}_p^0, \mathbb{C}_p^1) \cdot p.$$

Here, $\mathbb{C}_p^0$ and $\mathbb{C}_p^1$ mean the fiber over p of the trivial line bundles over L_0 and L_1, respectively. Of course, $\hom(\mathbb{C}_p^0, \mathbb{C}_p^1) = \mathbb{C}$ but it's good to remember what it means. Let $u : \mathbb{R} \times [0, 1]$ be a holomorphic strip from p to q (both in $L_0 \cap L_1$) with the proper boundary conditions: $u(\cdot, i) \in L_i$ and $\lim_{s\to\infty} u(s, t) = q$ (similar for p). Then, let $P(u)$ be the parallel transport along the loop from p to q and back which follows the boundary of u; while in L_0, we use ∇_0 and then we use ∇_1 in L_1. Then, the differential will be:

$$\partial p = \sum_{\operatorname{ind}(u)=1} \operatorname{sgn}(u) P(u) \cdot q$$

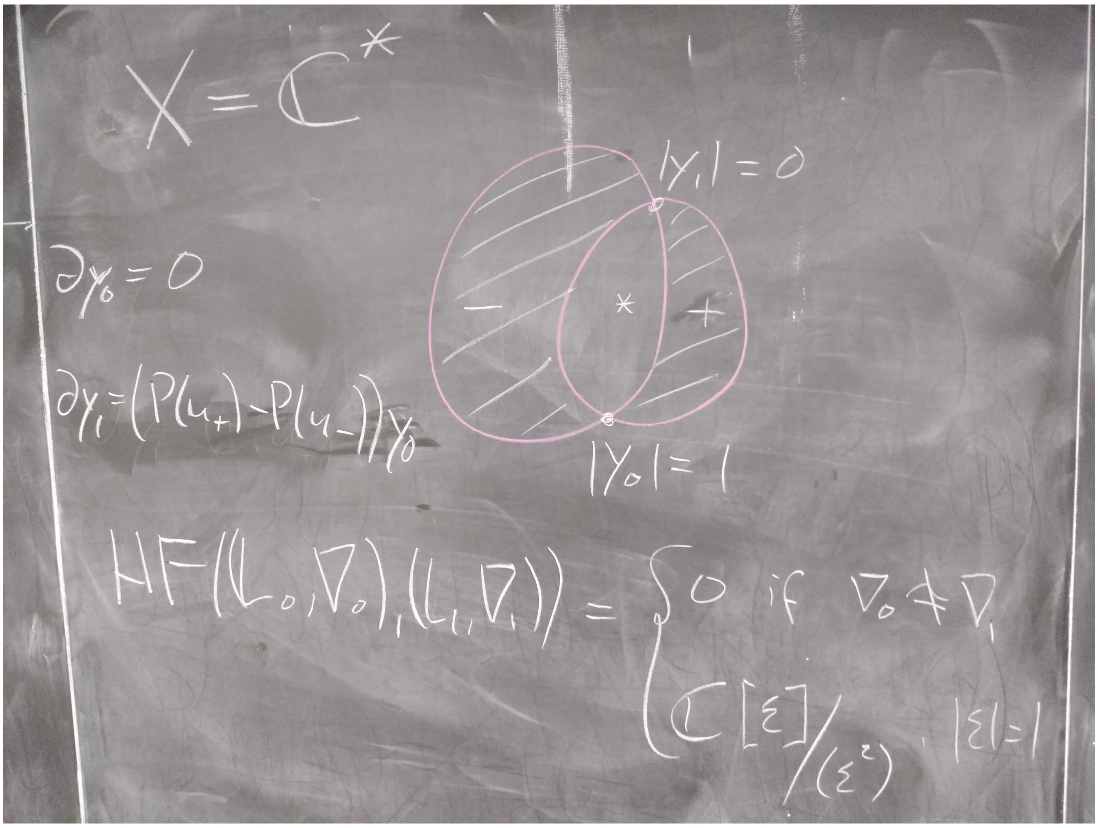
The motivation comes from an SYZ picture. In brief, suppose we have a special Lagrangian fibration $X \to B$ over a base. I think Calabi-Yau's are examples. Let $\check{X} := \{(L_b, [\nabla_b])\}$ where L_b is the fiber over b and $[\nabla_b]$ is a flat connection on L_b up to gauge equivalence. As it turns out, this forms a Kähler manifold; the holomorphic coordinates can be given by

$$Z_A(L_b, \nabla) = \exp\left(-\int_A \omega\right) \operatorname{hol}_\nabla(\partial A), A \in H_2(X, L, \mathbb{Z})$$

So $\check{X}$ is the mirror to X and the functor we want to define should send objects $(L_b, \nabla) \mapsto \mathcal{O}_{(L_b,\nabla)}$, a skyscraper sheaf.

Here is an example with $X = \mathbb{C}^*$. Consider two circles around the puncture that intersect in two points: y_0, y_1. They are Hamiltonian isotopic. The degrees of the points are $|y_0| = 1, |y_1| = 0$ and there is a unique strip u_+ with the boundary conditions as depicted (Riemann Mapping Theorem). Similarly, there is a unique u_-. However, $\partial y_0 = 0$ since there are no degree 2 points. Then, suppose the Hamiltonian isoptopy from L_0 to L_1 is given by φ. If $\varphi^*\nabla_1 \neq \nabla_0$, then the Floer homology is 0. Otherwise, it is a two dimensional thing, generated by y_0 and y_1. Mohamed chose to write this as $\mathbb{C}[\epsilon]/(\epsilon^2)$ to show the product structure. y_1 is degree 0, y_0 is degree 1 but there are no higher degree things so $y_0 \cdot y_0 = 0$. One could ask, "Why is $|y_0| = 1$ and $|y_1| = 0$? The picture is symmetric so shouldn't the degrees be symmetric and thus equal?"

It's a bit subtle but $CF(L_0, L_1)$ and $CF(L_1, L_0)$ are both generated by y_0, y_1 but the degrees change depending on which one we look at because it matters which direction we're going.



Two Hamiltonian isotopic Lagrangians in $\mathbb{C}^*$

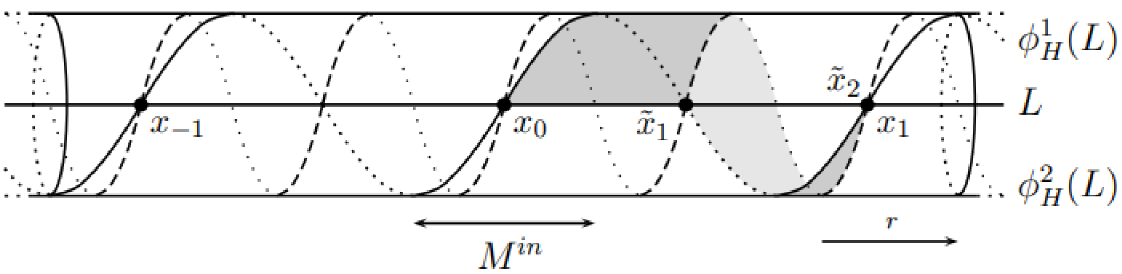
Moreover, we can see that with $X = \mathbb{C}^*$, its mirror, $\check{X}$, is also $\mathbb{C}^*$. To see this, think of $\mathbb{C}^* \cong T^*S^1 = \mathbb{R} \times S^1$, with coordinates (s, t). A flat connection is basically determined by its monodromy any so we have an S^1's worth of flat connections which we can parametrize with t. So in our enriched Fukaya category with local systems, we can consider a point $(s, e^{-2\pi i t})$ where s really means the Lagrangian fiber L_s and $e^{-2\pi i t}$ is a flat connection. Let $A \in H_2(T^*S^1, L_s, \mathbb{Z})$ be the section of the cylinder which is from 0 to s. It has area $2\pi s$. Using the holomorphic coordinates defined above, we have that $Z_A(s, e^{-2\pi i t}) = e^{-2\pi s} \cdot e^{-2\pi i t} = e^{-2\pi(s+it)} \in \mathbb{C}^*$. From this, we should guess that the mirror really is $\mathbb{C}^*$.

But we run into a few problems. If we have these compact Lagrangians L_s correspond to skyscraper sheaves $\mathcal{O}_p$, what mirrors the structure sheaf $\mathcal{O}_X \in \mathrm{ob}(D^b_{coh}(\mathbb{C}^*))$? We can see that $\hom(\mathcal{O}_X, \mathcal{O}_p) = \mathbb{C}$; something 1 dimensional. Thus, on the symplectic side, we want a Lagrangian that intersects each L_s exactly once. A good candidate is a (noncompact) cotangent fiber, call it $L \cong \mathbb{R}$. So we should add L into our considerations. I don't think it's so important that there are many choices of L. HMS doesn't say there is a unique functor; here, we evidently have at least an S^1's worth of functors since we have at least that many choices of L to map to $\mathcal{O}_X$.

There are now some other considerations. We notice that if we do a few Dehn twists to L, the result is Hamiltonian isotopic to L. But the number if intersection points depends on the number of twists! This is bad because it shows that in this noncompact case, $HF(L, L, H)$ depends on the choice of Hamiltonian H. What H should we choose if we're forced to choose one?

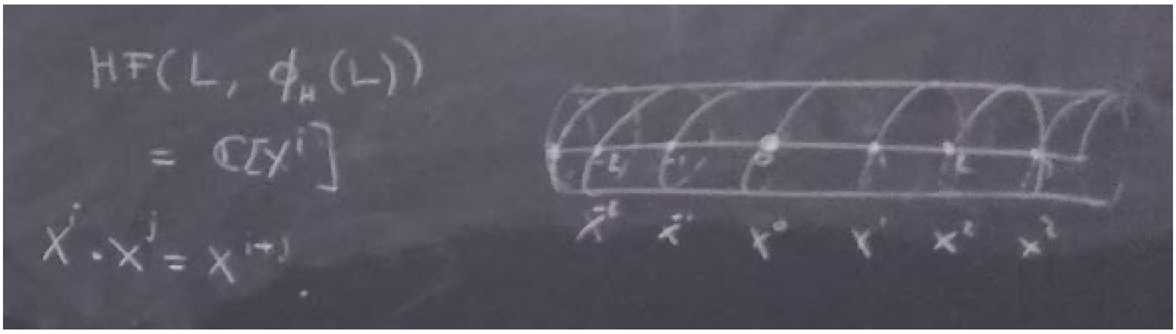
We can look at the complex side to get some advice. We note that $\hom(\mathcal{O}_X, \mathcal{O}_X) = \mathbb{C}[x, x^{-1}]$ which is infinite dimensional. As an aside, I believe that $\operatorname{Spec} \mathbb{C}[x, x^{-1}] = \mathbb{C}^*$. So we want $HF(L, L)$ to be infinite dimensional as well. For example, if we let $H = s^2$ at infinity, then the

Hamiltonian looks something like sR where R is the Reeb vector field scaled by s. This natural in light of contact geometry but there the important part is that this wraps L around so that $\varphi_H(L)$ intersects L infinitely many times. There are other ways to go about this process as well. We can choose Hamiltonians that wrap finitely many times (in both directions) and then take a direct limit: $HF(L,L) = \varinjlim_H HF(L, \varphi_H(L))$.



From Auroux's notes; ignore most of the notation

Having done this, we find that $HF(L,L) = \mathbb{C}[..., x_{-2}, x_{-1}, x_0, x_1, x_2, ...]$ and the product structure is $x_i \cdot x_j = x_{i+j}$. You can sort of see this product structure in the picture above with the shaded in region (which is a triangle wrapping around the cylinder).



Wrapping L