Notes on G_2 Manifolds

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These are notes taken from the first introductory talk on G_2 structures by Jin-Cheng Guu, in a seminar on gauge theory and G_2 manifolds. It is part of a larger project at the Simons Center to understand manifolds of special holonomy.

1 Cross Product Spaces

It is well known that the only division algebras which are vector space isomorphic to $\mathbb{R}^n$ are $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.

Now, the cross product on $\mathbb{R}^3$ seems to be peculiar in that the cross product is not defined on every $\mathbb{R}^n$. However, an observation about quaternion multiplication reveals that we can write, say $q = a + bi + cj + dk$ with a real (scalar) part and a vector (imaginary) part. Then, multiplication of quaternions can be represented by inner product and cross products.

Similarly, we can do this with octonions. There is a standard inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^8$ and we would like to define a cross product on $\mathbb{O}$. What properties do we want? We want skew-symmetry and $v \times w$ should be orthogonal to v and w. Also, we want $|v \times w|$ to be the area of the parallelogram spanned by v and w.

Let's take a hint from $\mathbb{H}$ to define a cross-product on $\mathbb{O}$. If $p = a + \vec{v}$ and $q = b + \vec{w}$ are quaternions represented with real part written as a scalar and imaginary part written as a vector, then $pq = (ab - \langle \vec{v}, \vec{w} \rangle) + a\vec{w} + b\vec{v} + \vec{v} \times \vec{w}$. Say, p, q are purely imaginary. Then this reduces to $pq = -\langle \vec{v}, \vec{w} \rangle + \vec{v} \times \vec{w}$. Okay, but now $p = \vec{v}, q = \vec{w}$ so we have $p \times q = pq + \langle p, q \rangle$.

This will be our definition for the cross product for imaginary octonions $V := \operatorname{Im} \mathbb{O} \cong \mathbb{R}^7$ (as vector spaces). It can be shown that it satisfies the properties we want. Let $\phi \in \Lambda^3 V^*$ be **the associated calibrated 3-form**. It is defined by $\phi(X, Y, Z) = \langle X \times Y, Z \rangle$.

2 Definition of G_2

Our first definition of G_2 is as follows:

Definition 2.1 (1st). *$G_2 \subset GL(V)$ is the subgroup preserving ϕ. That is, $G_2 = \{g \in GL(V) : g^*\phi = \phi\}$.*

We can write ϕ in standard coordinates; if we let $e_1, ..., e_7$ be the usual coordinates of $V \cong \mathbb{R}^7$ and ω^i be the dual of e_i, we can write the ϕ, now denoted as $\phi_0 = \omega^{123} + \omega^{145} + \omega^{167} + \omega^{246} - \omega^{257} - \omega^{347} - \omega^{356}$. Here, $\omega^{ijk} = \omega^i \wedge \omega^j \wedge \omega^k$. The indices may seem mysterious but there is some reason coming from how octonions multiply. Let's consider a second definition

Definition 2.2 (2nd). *Let $G_2 = Aut(\mathbb{O})$. That is, an element of G_2 is an invertible linear transformation A of $\mathbb{O}$ such that, for any $x, y \in \mathbb{O}$, $A(xy) = A(x)A(y)$. So it's an $\mathbb{O}$-algebra isomorphism.*

This is a natural definition. We now look at the third definition relates G_2 and $Spin(7)$.

Note that $\operatorname{Im}\mathbb{O}$ naturally acts on $\mathbb{O}$ by left multiplication but this is not associative. However, $v \cdot (v \cdot w) = (v \cdot v) \cdot w = -|v|^2 \cdot w$. This is a familiar Clifford algebra relation; thus, we may extend this action to $Cl(7)$, the Clifford algebra of $\mathbb{R}^7$.

Let's restrict our action now to $Spin(7) \subset Cl(7)$. So $Spin(7)$ acts on $\mathbb{O}$ and moreover, $S^7 \subset \mathbb{O}$ so $Spin(7)$ acts on S^7. The claim is, the action is **transitive.** Let $\psi \in S^7$ be **any** element. Then let $G_2 = \mathrm{Stab}(\psi)$.

Definition 2.3 (3rd)**.** *Let $G_2 = Stab(\psi)$, following the discussion above.*

The claim is that these definitions are all equivalent and moreover, G_2 is a simply connected, compact, real Lie group of dimension 14. This group is the smallest of the exceptional Lie groups and is isomorphic to the subgroup of $Spin(7)$ that preserves **any** chosen particular vector in its 8-dimensional real spinor representation. We'll prove a few of these claims. However, the last definition naturally leads us to consider some fiber bundles.

3 Three Fiber Bundles

The third definition basically tells us that we have the following fiber bundle:

$$\begin{tikzcd} G_2 \arrow[r, hook] & Spin(7) \arrow[d] \\ & S^7 \end{tikzcd}$$

We know $\dim Spin(7) = \dim SO(7) = \binom{7}{2} = 21$. So then $\dim G_2 = 14$. We can also see that G_2 acts on $\mathbb{R}^7$ and acts transitively on $S^6 \subset \mathbb{R}^7$. If $u \in S^6$, then, the complement in all of $\mathbb{R}^7$ is $u^\perp \cong \mathbb{R}^6$ and $SU(3)$ acts on $\mathbb{R}^6$ (somehow there is a Kähler structure on $\mathbb{R}^6$ that's naturally related to the ambient space). Anyways, the point is, we get another fibration:

$$\begin{tikzcd} SU(3) \arrow[r, hook] & G_2 \arrow[d] \\ & S^6 \end{tikzcd}$$

And lastly, we get a fibration by the inclusion $SU(2) \subset SU(3)$:

$$\begin{tikzcd} SU(2) \arrow[r, hook] & SU(3) \arrow[d] \\ & S^5 \end{tikzcd}$$

It's easy from the long exact sequence of homotopy groups on fibrations to see that $\pi_1(G_2) = \pi_2(G_2) = 0$; the second fact is true for all compact Lie groups and the fibrations do show that G_2 is compact. An interesting observation is that from one of the long exact sequences, we have $\pi_3(SU(3)) \cong \pi_3(SU(2)) = \pi_3(S^3) = \mathbb{Z}$. Another sequence gives $\pi_3(G_2) \cong \pi_3(SU(3)) = \mathbb{Z}$. Jin claims that all compact Lie groups with $\pi_3 = \mathbb{Z}$ are simple Lie groups. That is, it has no proper normal subgroup that are also Lie subgroups. He also said some stuff about Dynkin diagrams and root systems that looks interesting but I didn't understand it.

4 Topological G_2 Structures

Let M be a compact smooth 7-manifold. We know what a principal G-structure on the frame bundle F of M is. We just need the following commutative diagram:

$$\begin{tikzcd} G \arrow[r, hook] \arrow[d] & GL(7,\mathbb{R}) \arrow[d] \\ P_G \arrow[r, hook] \arrow[dr] & F \arrow[d] \\ & M \end{tikzcd}$$

So a topological G_2 structure is just defined this way. Here, we have a theorem which is not that impressive because we have a chain of subgroups and thus, we of course have the structures.

Theorem 4.1. *Having a $SU(2)$ structure implies we have an $SU(3)$ structure which implies we have a G_2 structure which implies we have a $Spin(7)$ structure. So then M is, in particular, spin.*

What is interesting is that we can actually show that $Spin(7)$ structure implies we have an $SU(2)$ structure! The proof relies on knowing that on any orientable 7-manifold M, there are two linearly independent vector fields X and Y. I don't know why; the Euler characteristic vanishes so we are guaranteed one at least. Let Δ_7 be the associated vector bundle to the $Spin(7)$ bundle with adjoint representation. It is of rank 8. Choose a spinor ψ; then $\psi, X\psi, Y\psi$ somehow construct for us a $SU(2)$ structure. So here's the theorem:

Theorem 4.2. *M has a $SU(2)$ structure if and only if it has a $SU(3)$ structure if and only if it has a G_2 structure if and only if it has a $Spin(7)$ structure.*

5 Geometric G_2 Structure

Suppose that ϕ is a 3-form on M which is nondegenerate everywhere. Then ϕ gives us a Riemannian metric g_ϕ which allows us to define a cross product on the bundle. We also have the Levi-Civita connection ∇^ϕ and Hodge-*. A natural question to ask is if ϕ is parallel with respect to ∇^ϕ; i.e. $\nabla^\phi \phi = 0$? In general, the answer is no but if the answer is yes, we'll call ϕ a **geometric G_2 structure**.

In other texts, a G_2 structure is a principal subbundle of the grame bundle of M with structure group G_2. Here, we're taking that to be the topological G_2 structure. Note that when we do have this principal G_2 bundle, then we automatically have the 3-form ϕ and metric g such that each tangent space of M admits an isomorphism to $\mathbb{R}^7$ which identifies ϕ with ϕ_0 and g with g_0.

Proposition 5.1. *$Hol_{\nabla^\phi}(M) \subset G_2$ if and only if $\nabla^\phi \phi = 0$ if and only if $d\phi = d(*\phi) = 0$.*

The proof of showing the holonomy impies the parallel condition is not probably not too hard. And showing the parallel condition implies the holonomy condition is also not too hard. But showing the equivalence of the parallel condition with the closed and co-closed conditions is not so easy. On one side, you have only one equation while you have two equations on the other. However, you have to really look at the octonion structure and you'll see that the $d\phi = 0$ and $d(*\phi) = 0$ conditions are actually related more closely than you think and in some sense, it's like having 1.5 equations instead of 2.

Proposition 5.2. *If M^7 admits geometrical G_2 structure , then M is Ricci flat.*

The proof of this seems to use the claim that $\nabla^\phi \phi = 0$ if and only if $\nabla^\phi \psi = 0$ for some $\psi \in \Gamma(\Delta_7)$. Now for some interesting remarks:

1. Rodrigo says that people often start with ϕ that satisfies $d(*\phi) = 0$ and tries to deform ϕ so that this property is maintained and eventually $d\phi = 0$.

2. Rodrigo also says that in the case of Kähler manifolds, if we know X holonomy is in $U(n)$ and we know $c_1(X) = 0$, then automatically, the holonomy is in $SU(n)$ and so X is Calabi-Yau. But there is no such theorem for G_2 manifolds and this is one of the difficulties of the subject.

6 Additional Facts about G_2 and $Spin(7)$ Manifolds

This section is taken from *Calabi-Yau Manifolds and Related Geometries*, specifically from the chapters written by Dominic Joyce.

6.1 More on G_2

Definition 6.1. *Joyce refers to a G_2 structure as a pair (φ, g) similar to what we have above. (φ, g) is* ***torsion-free*** *if $\nabla\varphi = 0$. From proposition 5.1, this means $Hol(g) \subset G_2$.*

Theorem 6.2. *Let (M, φ, g) be a compact G_2 manifold. Then $Hol(g) = G_2$ if and only if $\pi_1(M)$ is finite.*

Theorem 6.3. *Let M^7 be a compact 7-manifold, $\mathcal{X}$ the family of torsion-free G_2 structures (φ, g) on M, and $\mathcal{D}$ the group of diffeomorphisms of M isotopic to the identity. Then $\mathcal{M} = \mathcal{X}/\mathcal{D}$ is a smooth manifold of dimension $b_3(M)$ and the projection $\pi : \mathcal{M} \to H^3(M, \mathbb{R})$ given by $(\varphi, g)\mathcal{D} \mapsto [\varphi]$ is a local diffeomorphism.*

This theorem tells us that this moduli space of G_2 structures is a smooth manifold; it's not so strange that the dimension depends on b_3 since the φ's are 3-forms. Also, the tangent spaces of $\mathcal{M}$ are then modeled on H^3 because of the local diffeomorphism.

6.2 $Spin(7)$ Manifolds

Using the previous conventions, $\mathbb{R}^8$ has a 4-form $\Omega_0 = \omega^{1234} + \omega^{1256} + \omega^{1278} + \omega^{1357} - \omega^{1368} - \omega^{1458} - \omega^{1467} - \omega^{2358} - \omega^{2367} - \omega^{2457} + \omega^{2468} + \omega^{3456} + \omega^{3478} + \omega^{5678}$. I don't know what the pattern is. The subgroup of $GL(8, \mathbb{R})$ preserving this is $Spin(7)$; it also preserves the orientation of $\mathbb{R}^8$ and the Euclidean metric g_0. It is a compact, semisimple, 21 dim Lie group. Moreover, it is a subgroup of $SO(8)$.

Definition 6.4. *An 8-manifold M admits a $Spin(7)$* ***structure*** *if it has a principal subbundle of the frame bundle with structure group $Spin(7)$.*

A $Spin(7)$ manifold automatically gives a 4-form Ω and metric g; on each $x \in M$, there is an isomorphism (for that x) of $(T_xM, \Omega, g) \cong (\mathbb{R}^8, \Omega_0, g_0)$. Similar to before, the above definition is something of a topological definition while the pair (Ω, g) is a geometric $Spin(7)$ structure. We'll just use the words "$Spin(7)$ structure" to mean the pair (Ω, g).

Definition 6.5. *(M^8, Ω, g) is a $Spin(7)$ manifold if (Ω, g) is a $Spin(7)$ structure and $\nabla\Omega = 0$.*

Proposition 6.6. *Let (M^8, Ω, g) be a Spin(7) manifold. The following are equivalent:*

1. *$Hol(g) \subset Spin(7)$ with Ω as the induced 4-form.*

2. *$\nabla\Omega = 0$ on M; this is called **torsion-free**.*

3. *$d\Omega = 0$.*

When these hold, then g is Ricci-flat.

Theorem 6.7. *Let (M^8, Ω, g) be a compact Spin(7) manifold. Then $Hol(g) = Spin(7)$ if and only if $\pi_1(M) = 0$ and $b_3 + b_4^+ = b_2 + 2b_4^- + 25$.*

This is quite an interesting topological condition. I imagine there is some Atiyah-Singer machinery going on.

Theorem 6.8. *Let M^8 be a compact 8-manifold, $\mathcal{X}$ the family of torsion-free Spin(7) structures (Ω, g) on M, and $\mathcal{D}$ the group of diffeomorphisms of M isotopic to the identity. Then $\mathcal{M} = \mathcal{X}/\mathcal{D}$ is a smooth manifold of dimension $\hat{A}(M) + b_1 + b_4^-$ and the projection $\pi : \mathcal{M} \to H^4(M, \mathbb{R})$ given by $(\Omega, g)\mathcal{D} \mapsto [\Omega]$ is an immersion.*

Proposition 6.9. *If M admits Spin(7) structure, then $24\hat{A}(M) = -1 + b_1 - b_2 + b_3 + b_4^+ - 2b_4^-$.*

An immediate corollary to these two results is that if $\mathrm{Hol}\,(g) = Spin(7)$, then $b_1 = 0$ and so $\hat{A}(M) = 1$ and $\dim \mathcal{M} = 1 + b_4^-$.

The following commutative diagram shows the relationship between Calabi-Yaus, G_2, and $Spin(7)$ manifolds.

$$\begin{tikzcd}
SU(2) \arrow[r] \arrow[d] & SU(3) \arrow[r] \arrow[d] & G_2 \arrow[d] \\
SU(2) \times SU(2) \arrow[r] & SU(4) \arrow[r] & Spin(7)
\end{tikzcd}$$

Claim: If X is a Calabi-Yau 3-fold, then $\mathbb{R} \times X$ and $S^1 \times X$ have torsion-free G_2 structure.

6.3 Joyce's Construction of a Compact Manifold with $\mathrm{Hol}\,(g) = G_2$

Note, this is not simply a G_2 manifold; i.e. $\mathrm{Hol}\,(g) \subset G_2$, but $\mathrm{Hol}\,(g) = G_2$. Note that by our theorem above, if the holonomy is G_2, then $|\pi_1(M)| < \infty$. The construction is a Kummer type construction.

1. Let (φ_0, g_0) be a flat G_2 structure on T^7. Choose a finite group Γ of isometries preserving (φ_0, g_0). Then T^7/Γ is a compact 7 dim orbifold.

2. For certain Γ, the singularities are modeled on $\mathbb{R}^3 \times \mathbb{C}^2/G$ or $\mathbb{R} \times \mathbb{C}^3/G$ where G is a finite subgroup of $SU(2)$ or $SU(3)$. We resolve the singularities using complex geometric tools. Replace the singularities by $\mathbb{R}^3 \times X$ or $\mathbb{R} \times Y$ where X, Y are crepant resolutions of $\mathbb{C}^2/G$ or $\mathbb{C}^3/G$.

3. Now, we have a nonsingular compact 7-manifold M together with the resolving map $\pi : M \to T^7/\Gamma$. We can choose Γ such that $\pi_1(M)$ is finite.

4. Let (φ_t, g_t), $t \in (0, \epsilon)$ be a 1-parameter family of G_2 structures on M. They are not necessarily torsion-free but have small torsion when t is small. This family is defined using Calabi-Yau metrics on X, Y. The metrics satisfy asymptotic conditions at ∞ in X, Y. As $t \to 0$, $(\varphi_t, g_t) \to \pi^*(\varphi_0, g_0)$, a singular G_2 structure.

5. Deform (φ_t, g_t) to $\tilde{\varphi}_t, \tilde{g}_t)$ without torsion. Then since $\pi_1(M)$ is finite, $\tilde{g}_t$ is a metric with $\mathrm{Hol}\,(\tilde{g}_t) = G_2$.

Joyce constructs compact 8-manifolds with holonomy equal to $Spin(7)$ in a similar fashion.