Condensed Comprehensive Exams Study Sheet

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1 Complex Analysis

Definition 1.1. *A **harmonic function** is a twice continuously differentiable function $f : U \to \mathbb{R}$ (where U is an open subset of $\mathbb{R}^n$) which satisfies Laplace's equation, i.e.*

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0$$

everywhere on U. This is usually written as $\nabla^2 f = 0$.

Some basic facts about harmonic functions:

- The real and imaginary part of holomorphic functions are harmonic.
- **Maximum Principal**: If K is a nonempty compact subset of U, then f restricted to K attains its maximum and minimum on the boundary of K.
- **Mean Value Principal**: If Ω is open and $B(z,r) \subset \Omega \subset \mathbb{C}$, then the value $f(z)$ of a harmonic function $f : \Omega \to \mathbb{C}$ is given by the average value of f on the surface of the ball; this average value is also equal to the average value of f in the interior of the ball. So

$$f(z) = \frac{1}{\pi r^2} \int_{B(z,r)} f(w)\, dw.$$

Definition 1.2. *A **Möbius transformation** of $\mathbb{C}$ is a rational function of the form*

$$f(z) = \frac{az+b}{cz+d}$$

with $a, b, c, d \in \mathbb{C}$ satisfying $ad - bc \neq 0$.

Geometrically, a Möbius transformation can be obtained by stereographically projecting from the plane to S^2, rotating and moving S^2 to a new location and orientation in space, and then performing stereographic projection from the new position to the plane.

Möbius transformations are conformal; i.e. preserve angles. They take lines to circles or lines and circles to circles or lines. For every circle or line, there is a Möbius transformation which fixes it.

Theorem 1.3 (**Rouché's Theorem**). *For any two holomorphic functions f and g inside some region K with closed contour ∂K, if $|g(z)| < |f(z)|$ on ∂K, then f and $f + g$ have the same number of zeros inside K, where **each zero is counted as many times as its multiplicity**.*

Note the strict inequality.

Theorem 1.4 (**Morera's Theorem**). *A continuous, complex-valued function f defined on an open set $D \subset \mathbb{C}$ that satisfies*

$$\oint_\gamma f(z)\,dz = 0$$

for every closed piecewise C^1 curve γ in D must be holomorphic on D.

Theorem 1.5 (**Liouville's Theorem**). *Every bounded entire function must be constant. That is, every bounded function holomorphic on all of $\mathbb{C}$ is constant.*

Lemma 1.6 (**Schwarz's Lemma**). *Let $D = \{z : |z| < 1\}$ in $\mathbb{C}$ and let $f : D \to D$ be a holomorphic map such that $f(0) = 0$. Then, $|f(z)| \leq |z| \forall z \in D$ and $|f'(0)| \leq 1$. Moreover, if $|f(z)| = |z|$ for some non-zero z or $|f'(0)| = 1$, then $f(z) = \lambda z$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.*

Note that if $|f'(0)| < 1$, then it can't be a rotation since, if f were a rotation, $f(z) = \lambda z \Rightarrow f'(\lambda) = \lambda \Rightarrow |f'(z)| = |\lambda| = 1$. In this case, $|f(z)| < |z|$.

Theorem 1.7 (**The Riemann Mapping Theorem**). *Let $U \subsetneq \mathbb{C}$ be non-empty, open, and simply connected. There exists a bijective holomorphic $f : U \to D$ whose inverse is also holomorphic. Here, D is the open unit disk.*

2 Real Analysis

Fact: Let A be an $n \times n$ matric. Then $\det e^A = e^{\operatorname{tr} A}$.
Fact: Within the radius of convergence of $f(x) = \sum^\infty f_n(x)$, we may integrate or differentiate term by term and the sum equals the integral or derivative of f.

Theorem 2.1 (Theorem in Baby Rudin). *Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a,b]$ $(a < b)$, such that $\{f_n(x_0)\}$ converges for some $x_0 \in [a,b]$. If $\{f_n'\}$ converges uniformly on $[a,b]$, then $\{f_n\}$ converges uniformly on $[a,b]$, to a function f and $f'(x) = \lim_{n\to\infty} f_n'(x)$ for $x \in [a,b]$.*

Theorem 2.2 (**The Monotone Convergence Theorem**). *If $\{f_n\}$ is a sequence in L^+ such that $f_j \leq f_{j+1}$ for all j, and $f = \lim_{n\to\infty} f_n (= \sup_n f_n)$, then $\int f = \lim_{n\to\infty} \int f_n$.*

Lemma 2.3 (**Fatou's Lemma**). *If $\{f_n\}$ is any sequence in L^+, then*

$$\int \liminf f_n \leq \liminf \int f_n.$$

Theorem 2.4 (**The Dominated Convergence Theorem**). *Let $\{f_n\}$ be a sequence in L^1 such that*

1. *$f_n \to f$ a.e.*
2. *There exists $g \in L^1 \cap L^+$ such that $|f_n| \leq g$ a.e. for all n.*

Then $f \in L^1$ and $\int f = \lim_{n\to\infty} \int f_n$.

Theorem 2.5 (5.6: **The Hahn-Banach Theorem**). *Let X be a real vector space, p a sublinear functional on X, M a subspace of X, and f a linear functional on M such that $f(x) \leq p(x)$ for all $x \in M$. Then there exists a linear functional F on X such that $F(x) \leq p(x)$ for all $x \in X$ and $F|_M = f$.*

Theorem 2.6 (5.9: **The Baire Category Theorem**). *Let C be a complete **metric** space.*

1. *If $\{U_n\}^\infty$ is a sequence of open dense subsets of X, then $\bigcap^\infty U_n$ is dense in X.*

2. *X is not a countable union of nowhere dense sets.*

Theorem 2.7 (5.10: **The Open Mapping Theorem**). *Let X, Y be Banach spaces. If $T \in L(X, Y)$ is surjective, then T is open.*

Theorem 2.8 (5.12: **The Closed Graph Theorem**). *If X, Y are Banach spaces and $T : X \to Y$ is a closed linear map, then T is bounded.*

Theorem 2.9 (5.13: **The Uniform Boundedness Principle**). *Suppose that X, Y are normed vector spaces and $\mathcal{A} \subset L(X, Y)$.*

1. *If $\sup_{T\in\mathcal{A}} \|Tx\| < \infty$ for all x in some nonmeager subset of X, then $\sup_{T\in\mathcal{A}} \|T\| < \infty$.*

2. *If X is a Banach space and $\sup_{T\in\mathcal{A}} \|Tx\| < \infty$ for all $x \in X$, then $\sup_{T\in\mathcal{A}} \|T\| < \infty$.*

Theorem 2.10 (5.12: **Riesz Representation Theorem**). *If H is a Hilbert space, then for every bounded linear functional $f \in H^*$, there exists $y \in H$ such that $f(x) = \langle x, y\rangle$ for all $x \in H$.*

3 Group Theory

Fact: in S_n, if g, g' are of the same permutation type, then there exists an $h \in S_n$ such that $hgh^{-1} = g'$.

Other useful facts:

- If $G/Z(G)$ is cyclic, then G is abelian. Also, $G/Z(G) \cong \mathrm{Inn}(G)$.
- Let H be a normal subgroup of G. Then $G/C_G(H) \cong K \leqslant \mathrm{Aut}(H)$.

Lemma 3.1. *If H, K are subgroups of G and $H \leqslant N_G(K)$, then HK is a subgroup and $HK = KH$. Moreover, $HK \cong H \times K$.*

Lemma 3.2. *Let G be a finite group.*

- *If $|G| = pq$ where p, q are primes, $p < q$, and $p \nmid q - 1$, then $G \cong \mathbb{Z}_{pq}$.*
- *If $|G| = p^2$ where p is prime, then G is abelian by the Class Equation.*

Theorem 3.3 (**Burnside's Theorem**). *If G is a group with order $p^a q^b$ where p, q are primes, $a, b \in \mathbb{Z} \cup \{0\}$, then G is solvable. This immediately implies that every finite non-abelian simple group has order divisible by at least three distinct primes.*

Theorem 3.4 (**Feit-Thompson Theorem**). *If G is a finite group of odd order and is simple, then $G = \mathbb{Z}_p$ for some prime p.*

Lemma 3.5. *Let N be a normal subgroup of G with order n. If*
$1 \longrightarrow N \xrightarrow{\gamma} G \xrightarrow{\alpha} \mathbb{Z}_d \longrightarrow 0$ *is a sequence and $\gcd(n, d) = 1$, then $G \cong N \rtimes \mathbb{Z}_d$.*

Proof. Note that the sequence above is exact iff the sequence splits iff $G \cong N \rtimes \mathbb{Z}_d$ iff there is a g such that $g^d = 1$ and $\alpha(g)$ generates $\mathbb{Z}_d$. So choose any $a \in G$ such that $\alpha(a) = 1$. Then $\alpha(a^d) = d = 0$. If the sequence is exact, $a^d \in \ker\alpha = \mathrm{Im}\,\gamma$ which means a^d corresponds to an element in N. We'll use the notation $a^d \in N$. Then let the order $|a^d| = k$; by Lagrange, $k \mid n$. If $\gcd(n, d) = 1$, then $\gcd(k, d) = 1$. This is because k divides n.
So let $g := a^k$. Then since $\gcd(k, d) = 1$, $\alpha(g) = k$ which generates $\mathbb{Z}_d$. Also, since the order of a^d is k, then $(a^d)^k = (a^k)^d = g^d = 1$. Thus, the sequence splits so $G \cong N \rtimes \mathbb{Z}_d$. □

We may use this lemma in the case when $N = \mathbb{Z}_n$ and even more particular, when n, d are primes.

4 Galois Theory

The Galois group of a polynomial $p(x)$ over a field F is the group of automorphisms of K, the splitting field of p over F which fix the base field F.

Nota bene: When computing Galois groups, be careful to check whether p is **irreducible**. For instance, though $x^4 + 4$ has no roots in $\mathbb{Q}$, it equals $(x^2 + 2x + 2)(x^2 - 2x + 2)$. Also, check whether the roots can be represented by each other. This may affect the automorphisms. For instance, if $\pm\alpha, \pm\beta$ are roots but $\beta = 1/\alpha$, then if an automorphism sends $\alpha \mapsto -\alpha$, then $\beta \mapsto -\beta$.

Here are some facts.

- An extension K/F is Galois iff K is the splitting field of some separable polynomial over F; i.e. the polynomial doesn't have repeated roots.
- $|\mathrm{Gal}(K/F)| = [K : F]$, i.e. the dimension of the vector space K over F. Also, $|\mathrm{Gal}(K/F)| \leq (\deg(p))!$ (factorial). This is because the "largest" Galois group of a nth degree polynomial is the symmetric group S_n.
- Galois automorphisms only permute the roots of irreducible polynomials. So if p, q are irreducible polynomials, $\mathrm{Gal}(pq) = \mathrm{Gal}(p) \oplus \mathrm{Gal}(q)$.
- If p is irreducible in F with α as a root, then $|\mathrm{Gal}(F(\alpha)/F)| = \deg p$.
- Normal subgroups of the Galois group correspond to subfields which are Galois extensions of F. For example, if $p(x) = (x^3 + 1)(x^2 - 2)$ over $\mathbb{Q}$, then there is a normal subgroup which corresponds to $\mathbb{Q}[\sqrt{2}]$.
- From above, an extension K/F is Galois iff K is the splitting field of some separable polynomial over F. So if E is an extension $F \subset E \subset K$ and has an element α but E does not contain all the roots of the minimal polynomial of α over F, then it is not a splitting field. Thus, it corresponds to a **non-normal** subgroup. If a Galois group G has a non-normal subgroup, then G is non-abelian.
- The lattice of subfields and the lattice of subgroups is reversed.
- $F(\sqrt{\alpha})$ is **quadratic** if $\alpha \in F$ and char $F \neq 2$. This implies that $\mathrm{Gal}(F(\alpha)/F)$ is $\mathbb{Z}_2$.
- $F(\sqrt{\alpha}, \sqrt{\beta})$ is **biquadratic** if $\alpha, \beta \in F$ but $\sqrt{\alpha}, \sqrt{\beta}, \sqrt{\alpha\beta} \notin F$ and char $F \neq 2$. This implies that $\mathrm{Gal}(F(\alpha)/F)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Here are some **Irreducibility Criteria**.

- Let F be a field and $p(x) \in F[x]$. Then $p(x)$ has a factor of degree one if and only if $p(x)$ has a root in F. Thus, polynomials of degree 2 or 3 are reducible iff they have a root in F.
- For quartics, after checking for roots, if there are no roots in the field, check if it's the product of two quadratics. If p is irreducible, usually one can see derive a contradiction by assuming that it is the product of two quadratics.
- **Rational Root Test**: Let $p(x) = a_n x^n + ... + a_1 x + a_0$ with integer coeffcients. If $r/s \in \mathbb{Q}$ is in lowest terms and r/s is a root of p, then $r \mid a_0$ and $s \mid a_n$.

- **Gauss' Lemma**: Let R be a UFD with field of fractions F and $p(x) \in R[x]$. If $p(x)$ is reducible in $F[x]$, it is reducible in $R[x]$. The contrapositive is usually more useful, particularly with $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$.
- Let I be a proper ideal in the integral domain R and let $p(x)$ be a nonconstant monic polynomial in $R[x]$. If $\overline{p}(x) \in (R/I)[x] \cong R[x]/I[x]$ cannot be factored in $(R/I)[x]$ into two polynomials of smaller degree, then $p(x)$ is irreducible in $R[x]$.
- **Eisenstein's Criterion**: Let P be a prime ideal of integral domain R and let $f(x) = x^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$ be a polynomial in $R[x]$. Suppose that $a_{n-1}, ..., a_1, a_0 \in P$ but $a_0 \notin P^2$. Then $f(x)$ is irreducible in $R[x]$. So in $\mathbb{Z}[x]$, if p is prime and divides all the a_i but $p^2 \nmid a_0$, then f is irreducible.

5 Topology

Some useful facts:

- A manifold M is unorientable if and only if M has a orientable double cover.
- $\pi : X \to X/G$, the quotient map, is a covering map iff G is properly discontinuous.
- Deck transformations do not fix points. If X is Hausdorff, G is finite, and elements of G do not fix points, then G is properly discontinuous.
- If $p : \widetilde{X} \to X$ is a covering map from the universal cover of X to X, then p is trivially a regular covering map; i.e. $p_*(\pi_1(\widetilde{X}))$ is a normal subgroup of $\pi_1(X)$. Then, $X \cong \widetilde{X}/G$. In general, if p is regular, this holds.
- The antipodal map $a : S^n \to S^n$ is orientation preserving if n is odd.
- **Comps Lemma**: If M, N are n-manifolds with M compact and N connected, then if $F : M \to N$ is a submersion or immersion, then F is a covering map. The proof really only requires a local homeomorphism but when we have a submersion/immersion and the dimensions of the spaces equal, then $dF_p : T_pM \to T_{F(p)}N$ is max (constant) rank and invertible for each $p \in M$. Thus, by the **Inverse Function Theorem**, F is a local diffeomorphism.

6 de Rham Cohomology

Fact: The de Rham cohomologies are real vector spaces and are **homotopy invariant**.

- If M is a manifold with dimension n, then $H^k(M) = 0$ when $k > n$.
- Some simple cohomologies:

$$H^k(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & k = 0; \\ 0, & \text{else} \end{cases}, \quad H^k(S^n) = \begin{cases} \mathbb{R}, & k = 0, n; \\ 0, & \text{else} \end{cases}$$

- For **smooth manifolds**: Let M be a smooth n-manifold. $H^0(M) = \mathbb{R}^k$ where k is the number of connected components of M.

$$H^n(M) = \begin{cases} \mathbb{R}, & M \text{ orientable and compact}; \\ 0, & M \text{ unorientable or non-compact} \end{cases}$$

- $H^k(X \sqcup Y) = H^k(X) \oplus H^k(Y)$.
- If $\partial M \neq \varnothing$ and for all k, $H^k(M) = H^k(M \setminus \partial M)$, then $H^n(M) = 0$.
- **Künneth's Formula**:

$$H^k(X \times Y) = \bigoplus_{i+j=k} (H^i(X) \otimes_{\mathbb{R}} H^j(Y)).$$

- **Mayer-Vietoris**: Suppose $M = U \cup V$ is a n-manifold. Then we have the following **exact** sequence:

$$0 \to H^0(U \cup V) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \to H^1(U \cup V) \to H^1(U) \oplus H^1(V) \to$$
$$\to H^1(U \cap V) \to ... \to H^n(U \cup V) \to H^n(U) \oplus H^n(V) \to H^n(U \cap V) \to 0.$$

Sometimes, take advantage of the Coker $\gamma = V/\text{Im } \gamma$ since Im $\gamma = \ker \alpha$ for the next map α in the sequence.

Example 6.1. Suppose that $X = U \cap V$ is a smooth connected manifold and U, V are open connected subsets. Suppose $U \cap V$ isn't connected. Then $H^0(U \cap V) = \mathbb{R}^k$ where $k \geq 2$ is the number of connected components. Then we have this exact sequence:

$$0 \longrightarrow H^0(U \cup V) \xrightarrow{\alpha} H^0(U) \oplus H^0(V) \xrightarrow{\beta} H^0(U \cap V) \xrightarrow{\gamma} H^1(U \cup V)...$$

which corresponds to $0 \longrightarrow \mathbb{R} \xrightarrow{\alpha} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\beta} \mathbb{R}^k \xrightarrow{\gamma} H^1(U \cup V)...$

Supppose $H^1(U \cup V) = 0$. Then $0 \longrightarrow \mathbb{R} \xrightarrow{\alpha} \mathbb{R}^2 \xrightarrow{\beta} \mathbb{R}^k \xrightarrow{\gamma} 0$ is exact.

Suppose $k > 2$. Since Im $\beta = \ker \gamma = \mathbb{R}^k$, β is surjective. But the dimensions are wrong so $H^1(U \cup V) \neq 0$. If $k = 2$, then β is injective so Im $\alpha = \ker \beta = 0$. So $\ker \alpha = \mathbb{R}$. But α must be injective since the sequence is exact. So again, $H^1(U \cup V) \neq 0$.

- Let S be sequence:

$$... \to H^0(X) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \to H^1(X) \to ...$$

Then $\chi(S) := \sum_{k=1}^n (-1)^k \dim V_k$. If S is exact, then $\chi(S) = 0$. So if S is the Mayer-Vietoris sequence, then

$$\begin{aligned}\chi(S) = &-\dim(H^0(U \cup V)) + \dim(H^0(U) \oplus H^0(V)) - \dim(H^0(U \cap V)) \\ &+ \dim(H^1(U \cap V)) - ... \pm H^n(U \cap V).\end{aligned}$$