What is this dx in $\int f(x)\,dx$?

Sam Auyeung

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My freshman year of college, one of my floor mates, Andrew White, an engineering student, would write out his calculus homework and evaluate integrals without the dx. For example,

$$\int_0^\pi \cos x.$$

It irked me that he did not write the dx but I didn't have a good answer for why he should include it. He said, "It doesn't mean anything; all it tells you is that you should integrate with respect to x which you already know based on context."I didn't think about this question for many years, only concluding that Andrew was monumentally incorrect and being an engineering student, had no appreciation for the rigors and subtleties of mathematics. But once I started to teach calculus myself, I realized that many students have no idea what the dx means.

1 The Riemann Integral

When we define a definite Riemann integral, we can think of it as finding the area under a curve. Or more precisely, since sometimes the curve goes under the x axis, signed area "under" the curve. How might we try defining such a thing? If we have an interval $[1, 5]$, we can try approximating, say, the area under $f(x) = 3x^2$, between 1 and 5. We can start by using a rectangle with base length $b = 4$ and height $h = f(1) = 3$. What we get is some percentage of the area but we're clearly missing a large chunk. We can instead, approximate the area by using a rectangle with $b = 4$ and $h = f(5) = 75$. Now we've overshot and have more area than we should. This simple construction shows how we can choose the left or right endpoints to determine the height of our rectangles and depending on our situation, one may undershoot or overshoot the actual area.

So what do we do? We split the interval $[1, 5]$ up into smaller pieces and use rectangles slightly more tailored to our situation. Say we split the interval into 4 pieces: $[1, 2]$, $[2, 3]$, $[3, 4]$, and $[4, 5]$. We can take four rectangles with base length $b = 1$ and heights $f(1), f(2), f(3), f(4)$ (left endpoints) or $f(2), f(3), f(4), f(5)$ (right endpoints). This will give better approximations and it's clear we can improve our approximation in this toy model by taking more rectangles that have skinnier bases. Then, in the limit, we have many rectangles with infinitesimally thin bases.

Some remarks:

1. This partitioning of the interval doesn't have to be diadic; i.e. splitting things in half and then in half again, repeatedly. We could, instead, split $[1, 2]$ into 100 pieces and then have just one rectangle for $[2, 5]$.

2. One can note that if we do not split evenly, then the limit might not actually give what we want. Say we do all our partitioning on $[1, 2]$ and on $[2, 5]$, we always use a single rectangle. Our limit will never give a good approximations. So the picture I described above of taking a limit is not quite accurate. Rather, we need to consider all possible partitions and take an infimum on those that overshoot and also supremum on those that undershoot the area and see if the results match. If so, then this will be our Riemann integral.

3. One often does choose equal partitioning however, and thus, we see something like so for a definition of integration:

$$\int_a^b f(x)\, dx = (b-a) \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} f(x_k).$$

   The $1/n$ is often denoted by Δx: change in x. So one sees that the dx is supposed to be "infinitesimal change in x" as we let $n \to \infty$.

This last remark tells us a bit of what we mean by dx. It measures infinitesimal changes in x. If we think of $dx = 1 \cdot dx$, then all our rectangles are of height 1 with base length being a small Δx. If we multiply dx by $f(x)$, now, we're rescaling each of the infinitesimal steps by $f(x)$. Put another way, if we zoom in a lot, then the function f just rescales the rectangles from having height 1 to having height $f(x)$.

2 Brief Remarks on Lebesgue Integration

But there is also another type of integration which came about when people asked how we can integrate a function such as

$$f(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ 1, & x \notin \mathbb{Q} \end{cases}.$$

The problem is that since the rationals and irrationals are both dense in $\mathbb{R}$, we can't really take rectangles in the appropriate fashion. Say we're looking at integrating f on the interval $[0, 1]$. We can have finer and finer rectangles which all have endpoints on irrationals and so then the supremum is 1. But we can also have them all with endpoints on rationals and now the infimum is 0 and does $0 \neq 1$.

Hence, the idea of measures and Lebesgue integration was introduced. This type of integration allows us to consider a much larger range of functions. But of course, it depends on what measure μ we use. Thus, when we write the integral, the dx is replaced by a more general μ. This again indicates the importance of dx; it can be thought of as a measure.

$$\int_{[0,1]} f\, d\mu.$$

3 A More Geometric Understanding

Another view of dx can be from that of differential forms. In multivariable calculus, we often see some integral like

$$\int\int\int x^2 + y^2 + z^2\, dx\, dy\, dz.$$

Here, the dy and dz should point to infinitesimal changes in y and z directions and so taken together, $dx\, dy\, dz$ is like partitioning $\mathbb{R}^3$ into tiny rectangular prisms and finding volume this way. Indeed, in the language of differential forms, we would write this as $dx \wedge dy \wedge dz$ and this would be called a volume form. As it turns out, if we change coordinates to say, spherical coordinates (r, θ, φ), the differential forms change nicely and the volume is unaffected in the end.

Thus, dx is but one of the many 1-forms on $\mathbb{R}$. To use more algebraic jargon, it is a section of the cotangent bundle of $\mathbb{R}$; i.e. considering the bundle $\pi : T^*\mathbb{R} \to \mathbb{R}$, a 1-form σ satisfies $\pi \circ \sigma = \mathrm{id}$. At each point x, $\omega(x)$ is a linear functional: $\sigma(x) : T_x\mathbb{R} \to \mathbb{R}$. In our special case where the cotangent bundle is trivial; a section is nothing other than a map from the base to the fiber: $\sigma : \mathbb{R} \to \mathbb{R}^* = \hom(\mathbb{R}, \mathbb{R}) = \mathbb{R}$. But geometrically, we could think of dx as a volume form for the 1-dimensional manifold $\mathbb{R}$.

4 Conclusion

I've given three different views of what this dx means in different but overlapping contexts:

1. Riemann integration
2. Measure theory
3. Differential forms in geometry

One concluding observation to make is that if I write

$$\int_S f\, dx\, dy,$$

this should mean I'm integrating over a 2-dimensional object, say a surface S. Thus, if we consider what Andrew White had written,

$$\int \cos x,$$

since there is no dx written, this integration should really mean that we're integrating over a 0-dimensional object. That would just be a discrete set of points. Let's say the set is finite: $A = \{x_1, ..., x_k\}$. In this case, $\int_A \cos x$ should be reinterpreted. We need to consider what the measure should be. The counting measure μ seems to be a natural choice. Then this should be written as:

$$\int_A \cos x\, d\mu = \sum_{i=1}^{k} \cos(x_i).$$

If $A = \{1/2^n\}_{n=1}^{\infty}$ is infinite and say, the function is $f(x) = x$, then the "integral"

$$\int_A x = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

And thus, we're talking about series and their convergence, a topic usually reserved for Calculus II. So I suppose what Andrew wrote, $\int \cos x$, isn't nonsense. But he thought he should evaluate this on a 1-dimensional space when it should mean evaluating on a 0-dimensional space.