

ON SURFACES IN \mathbb{P}^4 AND 3-FOLDS IN \mathbb{P}^5

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0. INTRODUCTION

We report on some recent progress in the classification of smooth projective varieties with small invariants. This progress is mainly due to the finer study of the adjunction mapping by Reider, Sommese and Van de Ven [So1], [VdV], [Rei], [SV]. Adjunction theory is a powerful tool for determining the type of a given variety. Classically, the adjunction process was introduced by Castelnuovo and Enriques [CE] to study curves on ruled surfaces. The Italian geometers around the turn of the century also started the classification of smooth surfaces in \mathbb{P}^4 of low degree. Further classification results are due to Roth [Ro1], who uses the adjunction mapping to get surfaces with smaller invariants already known to him (compare [Ro2] for adjunction theory on 3-folds). Nowadays, through the effort of several mathematicians, a complete classification of smooth surfaces in \mathbb{P}^4 and smooth 3-folds in \mathbb{P}^5 has been worked out up to degree 10 and 11 resp. Moreover, in the 3-fold case the classification is almost complete in degree 12. For references see section 7.

One motivation to study these varieties comes from Hartshorne's conjecture [Ha1]. In the case of codimension 2 this suggests that already smooth 4-folds in \mathbb{P}^6 should be complete intersections. Another motivation originates from two mutually corresponding finiteness results. Ellingsrud and Peskine [EP] proved that there are only finitely many families of smooth surfaces in \mathbb{P}^4 which are not of general type. However, the question of an exact degree bound d_0 is still open. By [BF] $d_0 \leq 105$. Examples are known only up to degree 15 and one actually believes that $d_0 = 15$. The analogous finiteness result holds for 3-folds in \mathbb{P}^5 [BOSS1]. In this case one expects a much higher degree bound. Nevertheless examples had been known so far only up to degree 14 [Ch3]. In this note we present, among other things, three new smooth 3-folds in \mathbb{P}^5 of degree 13, 17 and 18 resp.

How to construct examples ?

Let us recall that every smooth projective variety of dimension m can be embedded in \mathbb{P}^{2m+1} . So e.g. in the surface case we could try to work with general projections from points in \mathbb{P}^5 . However Severi's theorem [Se] tells us that every non-degenerate smooth surface in \mathbb{P}^4 except the Veronese surface is linearly normal. Similarly by Zak's theorem [Za] every non-degenerate smooth 3-fold in \mathbb{P}^5 is linearly normal.

There are two other classical construction methods. One is to study linear systems on abstract varieties. This works especially well for rational, abelian and bielliptic surfaces. The other is liaison [PS] starting with a known local complete intersection variety (presumably of lower degree). With a few exceptions these methods failed to produce examples in higher degree. In the case of liaison this is mainly due to the fact, that the varieties to be constructed tend to be minimal in their even liaison class (compare [LR]). Whereas, if we consider e.g. linear systems of curves on minimal surfaces, the base points have to be in a special position. Such configurations are hard to find.

In this context a new construction method for surfaces $X \subset \mathbb{P}^4$ (more generally $(n-2)$ -folds $X \subset \mathbb{P}^n$) was introduced in [DES] (compare also [Po]). The basic idea is an application of Beilinson's spectral sequence [Bei]: To construct the ideal sheaf \mathcal{I}_X and thus X one has to construct the Hartshorne-Rao modules of X first. Involving corresponding syzygy bundles as suggested by the spectral sequence one finds vector bundles \mathcal{F} and \mathcal{G} on \mathbb{P}^n with $\text{rk } \mathcal{G} = \text{rk } \mathcal{F} + 1$, and a morphism $\varphi \in \text{Hom}(\mathcal{F}, \mathcal{G})$, whose minors define the desired X . From the syzygies of the Hartshorne-Rao modules one can compute the syzygies of \mathcal{I}_X and so the explicit equations for X . Typically, part of the geometry behind X can already be seen from the syzygies. The smoothness of X can be checked via the implicit function theorem, i.e., by a straightforward computation. Since these computations are very extensive one has to rely on a computer and a computer algebra system. Currently, Macaulay [Mac] is the only system which is powerful enough to handle the computations.

In some cases X is not minimal in its even liaison class, or a minimal element in the complementary even liaison class has low degree and can be identified. In fact, by studying the equations we find examples where X can be reconstructed via liaison from a reducible scheme X' of lower degree. It is hard to find such reducible schemes a priori.

Problem. *Find a geometric construction for all examples constructed via syzygies.* \square

Notation. $R = \mathbb{C}[x_0, \dots, x_n] = \bigoplus_{q \in \mathbb{Z}} S^q V^*$ will be the homogeneous coordinate ring of \mathbb{P}^n , so $H^0(\mathcal{O}_{\mathbb{P}^n}(1)) = V^*$. If $X \subset \mathbb{P}^n$ is a fixed smooth subvariety, then d will denote its degree, π its sectional genus, H the hyperplane class and K the canonical class. \square

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1. CONSTRUCTIONS VIA SYZYGIES

Following [DES] we want to construct a codimension 2 subvariety $X \subset \mathbb{P}^n$ as the determinantal locus of a map between vector bundles. So we are looking for vector bundles \mathcal{F} and \mathcal{G} on \mathbb{P}^n with $\text{rk } \mathcal{F} = f$ and $\text{rk } \mathcal{G} = f + 1$, and a morphism $\varphi \in \text{Hom}(\mathcal{F}, \mathcal{G})$ whose minors vanish in the expected codimension 2. In this case $X = V(\varphi)$ is a locally Cohen-Macaulay subscheme and the Eagon-Northcott complex [BE]

$$0 \leftarrow \mathcal{O}_X(m) \leftarrow \mathcal{O}(m) \cong \bigwedge^f \mathcal{F}^* \otimes \bigwedge^{f+1} \mathcal{G} \leftarrow \mathcal{G} \xrightarrow{\varphi} \mathcal{F} \leftarrow 0$$

is exact and identifies $\text{coker } \varphi$ with the twisted ideal sheaf

$$\text{coker } \varphi \cong \mathcal{I}_X(m), \quad m = c_1 \mathcal{G} - c_1 \mathcal{F}.$$

Furthermore, a mapping cone between the minimal free resolutions of \mathcal{F} and \mathcal{G} is a (not necessarily minimal) free resolution of $\mathcal{I}_X(m)$. So for a given φ we can derive an explicit system of homogeneous equations for its dependency locus X .

Remark 1.1. Let $\varphi_1, \varphi_2 \in \text{Hom}(\mathcal{F}, \mathcal{G})$ be morphisms whose minors vanish in codimension 2. Then $V(\varphi_1)$ and $V(\varphi_2)$ lie in the same irreducible component of the Hilbert scheme (compare e.g. [BB], [BBM], [MDP]). \square

To construct a variety with the desired numerical invariants one has to find appropriate \mathcal{F} and \mathcal{G} . Clearly \mathcal{F} and \mathcal{G} reflect the structures of the graded finite length R -modules

$$H_*^i \mathcal{I}_X = \bigoplus_{q \in \mathbb{Z}} H^i(\mathbb{P}^n, \mathcal{I}_X(q)), \quad i = 1, \dots, \dim X,$$

called the Hartshorne-Rao modules of X . E.g., X is projectively Cohen-Macaulay, i.e., its Hartshorne-Rao modules are trivial, iff \mathcal{F} and \mathcal{G} can be chosen to be direct sums of line bundles. Or compare [Ch2] for the Ω -resolution of a projectively Buchsbaum variety. In this case, in particular, the multiplication maps of the Hartshorne-Rao modules are trivial.

Remark 1.2. Smooth projectively Cohen-Macaulay and smooth projectively Buchsbaum varieties of codimension 2, which are not of general type, are completely classified (compare [Ch3]). \square

In any case it is a natural idea to construct the Hartshorne-Rao modules first. Then one may involve corresponding syzygy bundles as direct summands in order to find \mathcal{F} and \mathcal{G} . Recall:

Proposition 1.3. Let $M = \bigoplus_{q \in \mathbb{Z}} M_q$ be a graded R -module of finite length and let

$$0 \leftarrow M \leftarrow L_0 \xrightarrow{\alpha_1} L_1 \leftarrow \dots \xrightarrow{\alpha_{n+1}} L_{n+1} \leftarrow 0$$

be its minimal free resolution. Then, for $1 \leq i \leq n - 1$, the sheafified syzygy module

$$\mathcal{F}_i = \text{Syzy}_i(M) = (\ker \alpha_i)^\sim = (\text{Im } \alpha_{i+1})^\sim$$

is a vector bundle on \mathbb{P}^n with the intermediate cohomology

$$\bigoplus_{q \in \mathbb{Z}} H^j(\mathbb{P}^n, \mathcal{F}_i(q)) = \begin{cases} M & j = i \\ 0 & j \neq i, \quad 1 \leq j \leq n-1 \end{cases} .$$

Conversely, any vector bundle \mathcal{F} on \mathbb{P}^n with this intermediate cohomology is stably equivalent with \mathcal{F}_i , i.e.,

$$\mathcal{F} \cong \mathcal{F}_i \oplus \mathcal{L}, \quad \mathcal{L} \text{ a direct sum of line bundles.} \quad \square$$

Example 1.4. Consider \mathbb{C} as a graded R -module sitting in degree 0. The minimal free resolution of $\mathbb{C}(i)$ is the Koszul complex

$$0 \leftarrow \mathbb{C}(i) \leftarrow \bigwedge^0 V^* \otimes R(i) \leftarrow \dots \leftarrow \bigwedge^{n+1} V^* \otimes R(i-n-1) \leftarrow 0 .$$

The corresponding syzygy bundles are the twisted bundles of differentials, $\text{Syzy}_i(\mathbb{C}(i)) \cong \Omega^i(i)$. It follows from the sheafified Koszul complex, that $\text{Hom}(\Omega^i(i), \Omega^j(j)) \cong \bigwedge^{i-j} V$, the isomorphisms being given by contraction (cf. [Bei]). \square

Which syzygy bundles should be involved in the construction of \mathcal{F} and \mathcal{G} ? This can be found out by analyzing Beilinson's spectral sequence for $\mathcal{J}_X(m)$. Recall:

Theorem 1.5. [Bei] *For any coherent sheaf \mathcal{S} on \mathbb{P}^n there is a spectral sequence with E_1 -terms*

$$E_1^{pq} = H^q(\mathbb{P}^n, \mathcal{S}(p)) \otimes \Omega^{-p}(-p)$$

converging to \mathcal{S} , i.e., $E_\infty^{pq} = 0$ for $p+q \neq 0$ and $\bigoplus E_\infty^{-p,p}$ is the associated graded sheaf of a suitable filtration of \mathcal{S} . \square

This theorem is often used to construct \mathcal{S} by determining the differentials of the spectral sequence first. A crucial point is that the d_1 -differentials

$$\begin{aligned} d_1^{pq} &\in \text{Hom}(H^q(\mathbb{P}^n, \mathcal{S}(p)) \otimes \Omega^{-p}(-p), H^q(\mathbb{P}^n, \mathcal{S}(p+1)) \otimes \Omega^{-p-1}(-p-1)) \\ &\cong \text{Hom}(V^* \otimes H^q(\mathbb{P}^n, \mathcal{S}(p)), H^q(\mathbb{P}^n, \mathcal{S}(p+1))) \end{aligned}$$

coincide with the natural multiplication maps. In our case $\mathcal{S} = \mathcal{J}_X(m)$, and we will interpret one part of Beilinson's spectral sequence as the spectral sequence of a vector bundle \mathcal{F} , the other part as that of a vector bundle \mathcal{G} . The differential between the two parts will give the morphism $\varphi: \mathcal{F} \rightarrow \mathcal{G}$ whose cokernel is the desired $\mathcal{J}_X(m)$. The twist m will be mainly n or $n-1$ (compare [DES, 1.7] for the corresponding Beilinson cohomology tables in the surface case).

How to check the smoothness of X ? If the bundle $\text{Hom}(\mathcal{F}, \mathcal{G})$ is globally generated and $n \leq 5$, then we know from [Klm], that the generic $\varphi \in \text{Hom}(\mathcal{F}, \mathcal{G})$ gives rise to a smooth X . This works well, if X is projectively Cohen-Macaulay. Similarly, if X is projectively Buchsbaum, we may apply [Ch1]. In the general case however, we mostly have to rely on a computer as explained in the introduction.

Example 1.6. We will construct a family of smooth 3-folds $X \subset \mathbb{P}^5$ with the numerical invariants $d = 18$, $\pi = 35$, $\chi(\mathcal{O}_X) = 2$ and $\chi(\mathcal{O}_S) = 26$, where S is a general hyperplane section of X . Let us analyze Beilinson's spectral sequence for $\mathcal{J}_X(4)$. We first need information on the dimensions $h^i \mathcal{J}_X(m)$, $m = -1, \dots, 4$. In view of Riemann-Roch a plausible Beilinson cohomology table is

24					
	1	6	3		

Suppose that a smooth 3-fold X with these data exists. Then Beilinson's theorem yields an exact sequence

$$0 \rightarrow \mathcal{F} = 24\mathcal{O}(-1) \rightarrow \mathcal{G} \rightarrow \mathcal{J}_X(4) \rightarrow 0,$$

where \mathcal{G} is the cohomology of a monad

$$0 \rightarrow \Omega^4(4) \xrightarrow{d_1^{-4,3}} 6\Omega^3(3) \xrightarrow{d_1^{-3,3}} 3\Omega^2(2) \rightarrow 0.$$

On the other hand, the generic module with Hilbert function $(1, 6, 3)$ has syzygies of type

$$\begin{array}{ccccccc}
 0 \leftarrow M \leftarrow R(4) & & & & & & \\
 & \swarrow & & & & & \\
 & & 18R(2) \leftarrow 52R(1) \leftarrow 60R & & 24R(-1) & & \\
 & & & & \oplus & & \\
 & & & & 10R(-2) & \swarrow & 12R(-3) \leftarrow 3R(-4) \leftarrow 0.
 \end{array}$$

A check on the ranks and the intermediate cohomology of \mathcal{G} and $\text{Syzy}_3(M)$ suggests that conversely it is promising to start with $\mathcal{F} = 24\mathcal{O}(-1)$ and $\mathcal{G} = \text{Syzy}_3(M)$. Indeed, for the map $\varphi \in \text{Hom}(24\mathcal{O}(-1), \text{Syzy}_3(M))$ given by the syzygies, one can check that the minors of φ vanish along a smooth 3-fold X . By construction \mathcal{J}_X has syzygies of type

$$0 \leftarrow \mathcal{J}_X \leftarrow 10\mathcal{O}(-6) \leftarrow 12\mathcal{O}(-7) \leftarrow 3\mathcal{O}(-8) \leftarrow 0.$$

In particular, X is cut out by 10 sextics. From the syzygies it follows that $\omega_X(1) = \mathcal{E}xt^2(\mathcal{O}_X, \mathcal{O}(-6))(1)$ is a quotient of $24\mathcal{O}$, and since $(K + H)^2 \cdot K = -4$ (compare section 5) we deduce that the Kodaira dimension $\kappa(X) = -\infty$. \square

2. LIAISON

We recall the definition and some basic results [PS]. Let $X, X' \subset \mathbb{P}^n$ be two locally Cohen-Macaulay subschemes of pure codimension 2 with no irreducible components in common.

X and X' are said to be (geometrically) *linked* (r, s) , if there exist hypersurfaces V_1 and V_2 of degrees r and s resp. such that $X \cup X' = V_1 \cap V_2$. Then there are the standard exact sequences

$$0 \rightarrow \omega_X \rightarrow \mathcal{O}_{V_1 \cap V_2}(r+s-n-1) \rightarrow \mathcal{O}_{X'}(r+s-n-1) \rightarrow 0,$$

$$0 \rightarrow \omega_X \rightarrow \mathcal{O}_X(r+s-n-1) \rightarrow \mathcal{O}_{X \cap X'}(r+s-n-1) \rightarrow 0.$$

The degrees and sectional genera of X and X' are related by

$$d + d' = r \cdot s \quad \text{and} \quad \pi - \pi' = \frac{1}{2}(r+s-4)(d-d'),$$

and

$$\chi(\mathcal{O}_{X'}) = \chi(\mathcal{O}_{V_1 \cap V_2}) - \chi(\mathcal{O}_X(r+s-n-1)).$$

Under suitable assumptions (e.g., if $H^1(\mathcal{F}(r)) = H^1(\mathcal{F}(s)) = 0$) we may deduce from a given locally free resolution $0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{J}_X \rightarrow 0$ of \mathcal{J}_X a resolution

$$0 \rightarrow \mathcal{G}(-r-s) \rightarrow \mathcal{F}(-r-s) \oplus \mathcal{O}(-r) \oplus \mathcal{O}(-s) \rightarrow \mathcal{J}_{X'} \rightarrow 0$$

of $\mathcal{J}_{X'}$ by taking a mapping cone as in [PS, Prop. 2.5]. Moreover, the Hartshorne-Rao modules of $\mathcal{J}_{X'}$ are \mathbb{C} -dual to those of X :

$$(H_*^{n-1-i} \mathcal{J}_X)^* \cong (H_*^i \mathcal{J}_{X'})(n+1-r-s), \quad i = 1, \dots, n-2.$$

Liaison can be used to construct new subvarieties starting from given ones. Hence it is useful to know, under which conditions a residual intersection will be smooth. One result in this direction is a special case of [PS, Prop.4.1]:

Theorem 2.1. (*Peskine-Szpiro*). *Let $X \subset \mathbb{P}^n$, $n \leq 5$, be a local complete intersection of codimension 2. Let m be a twist such that $\mathcal{J}_X(m)$ is globally generated. Then for every pair $d_1, d_2 \geq m$ there exist forms $f_i \in H^0(\mathcal{J}_X(d_i))$, $i = 1, 2$, such that the corresponding hypersurfaces V_1 and V_2 intersect properly, $V_1 \cap V_2 = X \cup X'$, where*

- (i) X' is a local complete intersection,
- (ii) X and X' have no common component,
- (iii) X' is nonsingular outside a set of positive codimension in $\text{Sing } X$. \square

3. ADJUNCTION THEORY

In this section (X, H) will denote a polarized pair, where X is a smooth, connected, projective variety of dimension $m \geq 2$ and H is a very ample divisor on X . $K = K_X$ will be a canonical divisor on X . Before reviewing the general theory behind the *adjunction* map $\Phi = \Phi_{|K+(m-1)H|}$, we will give an example.

Example 3.1. [Roo] Let $\varphi = (\varphi_{ij})_{\substack{0 \leq i \leq 2 \\ 0 \leq j \leq 3}}$ be a general 3×4 -matrix with linear entries in $\mathbb{C}[x_0, \dots, x_4]$. Then $X = V(\varphi)$ is a smooth surface $X \subset \mathbb{P}^4$ with $d = 6$ and $\pi = 3$. Let H be the hyperplane class of X . By dualizing φ , we obtain the resolution

$$0 \leftarrow \omega_X(1) \leftarrow 3\mathcal{O} \xleftarrow{\varphi} 4\mathcal{O}(-1) \leftarrow \mathcal{O}(-4) \leftarrow 0.$$

So $|K + H|$ is base point free, $N = \dim |K + H| = 2$ and we have a well-defined adjunction map $\Phi = \Phi_{|K+H|} : X \rightarrow \mathbb{P}^2$. Let y_0, y_1, y_2 be coordinates on \mathbb{P}^2 . Then $\text{graph}(\Phi) \subset \mathbb{P}^4 \times \mathbb{P}^2$ is given by the equations

$$y_0\varphi_{0j}(x) + y_1\varphi_{1j}(x) + y_2\varphi_{2j}(x) = 0, \quad j = 0, \dots, 3.$$

We may rewrite these equations as

$$x_0\psi_{j0}(y) + \dots + x_4\psi_{j4}(y) = 0, \quad j = 0, \dots, 3,$$

where $\psi = (\psi_{jk})_{\substack{0 \leq j \leq 3 \\ 0 \leq k \leq 4}}$ has linear entries in $\mathbb{C}[y_0, y_1, y_2]$. The general fibre of Φ is defined by four independent linear forms in $\mathbb{C}[x_0, \dots, x_4]$. Hence Φ is birational with positive dimensional fibres precisely in the points where ψ drops rank:

$$0 \rightarrow 4\mathcal{O}_{\mathbb{P}^2}(-5) \xrightarrow{\psi} 5\mathcal{O}_{\mathbb{P}^2}(-4) \rightarrow \mathcal{J}_Z \rightarrow 0.$$

So $\Phi : X \rightarrow \mathbb{P}^2$ expresses X as the blowing up of 10 points in \mathbb{P}^2 and X is embedded by the quartics through these points, i.e., by the 4×4 -minors of ψ . In other words

$$H \equiv 4L - \sum_{i=1}^{10} E_i$$

(with obvious notations) and X is a Bordiga surface. \square

The first general result deals with the existence of the adjunction map. It is a consequence of [So1], [VdV].

Theorem 3.2. *Let (X, H) and K be as above. Then $|K + (m - 1)H|$ is base point free unless*

- (i) $(X, \mathcal{O}_X(H)) \cong (\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1))$ or $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$,
- (ii) $(X, \mathcal{O}_X(H)) \cong (Q, \mathcal{O}_Q(1))$, where $Q \subset \mathbb{P}^{m+1}$ is a smooth hyperquadric,
- (iii) $(X, \mathcal{O}_X(H))$ is a scroll over a smooth curve. \square

If $|K + (m - 1)H|$ is base point free, then we denote by

$$\begin{array}{ccc} X & \xrightarrow{\Phi} & \mathbb{P}^N \\ r \searrow & & \nearrow s \\ & X' & \end{array}$$

the Stein factorization of the adjunction map Φ . X' is normal, r is connected and s is finite.

Theorem 3.3. [So2] *Let (X, H) and K be as above and suppose that $|K + (m - 1)H|$ is base point free. Then there are the following possibilities:*

- (i) $\dim \Phi(X) = 0$, and $K \equiv -(m - 1)H$, i.e., X is Fano of index $(m - 1)$,
- (ii) $\dim \Phi(X) = 1$, and the general fibre of r is a smooth quadric Q such that H induces $\mathcal{O}_Q(1)$,
- (iii) $\dim \Phi(X) = 2 < m$ and r exhibits X as a scroll over a smooth surface,
- (iv) $\dim \Phi(X) = m$. \square

If $\dim \Phi(X) = m$ we write $L' = r_*(H)$, $K' = K_{X'}$ and $H' = K' + (m-1)L'$. The next result tells us, that in this case r contracts precisely the linear $\mathbb{P}^{m-1} \subset X$ with normal bundle $\mathcal{O}_{\mathbb{P}^{m-1}}(-1)$ (necessarily disjoint).

Theorem 3.4. [So1],[So2] *Suppose that $\dim \Phi(X) = m$. Then $r : X \rightarrow X'$ is the blowing up of a finite number of points on the smooth projective variety X' . L' and H' are ample and*

$$r^*(H') \equiv K + (m-1)H . \quad \square$$

In the above situation (X', L') is called the *first reduction* of (X, H) [So5].

When is s an embedding? The answer is given by

Theorem 3.5. [SV] *Suppose that $\dim \Phi(X) = m$. Then H' is very ample, unless X is a surface and*

- (i) $X = \mathbb{P}^2(p_1, \dots, p_7)$ and $H \equiv 6L - \sum_{i=1}^7 2E_i$ (the Geiser involution),
- (ii) $X = \mathbb{P}^2(p_1, \dots, p_8)$ and $H \equiv 6L - \sum_{i=1}^7 2E_i - E_8$,
- (iii) $X = \mathbb{P}^2(p_1, \dots, p_8)$ and $H \equiv 9L - \sum_{i=1}^8 3E_i$ (the Bertini involution),
- (iv) $X = \mathbb{P}(\mathcal{E})$, where \mathcal{E} is an indecomposable rank 2 bundle over an elliptic curve, and $H \equiv 3B$, where B is a section with $B^2 = 1$ on X . \square

For surfaces the *adjunction process*, i.e., the study of $|K+H|$, $|K'+H'|$ etc., will finally lead to a minimal model. For 3-folds $X \subset \mathbb{P}^5$ the situation is quite different. In this case it is often successful to study $|K+H|$ instead of $|K+2H|$. Compare section 5 for details and applications of further general results of adjunction theory.

4. SURFACES IN \mathbb{P}^4

In this section X will denote a smooth non-degenerate surface in \mathbb{P}^4 and $d = H^2$ its degree, $\pi = \frac{1}{2}H \cdot (K+H) + 1$ its sectional genus and $\chi = \chi(\mathcal{O}_X) = 1 - q + p_g$ its Euler characteristic. K^2 may be computed from the double point formula (cf. [Ha2, Appendix A, 4.1.3.]

$$d^2 - 10d - 5H \cdot K - 2K^2 + 12\chi = 0 .$$

In order to classify surfaces of a given degree, one first has to work out a finite list of admissible numerical invariants. One may apply Halphen's upper bound for π [GP] in connection with the lifting theorem of Roth [Ro1, p.152] and the following classification results:

Theorem 4.1. [Ro1], [Au]. *Let X be contained in a hyperquadric $V^2 \subset \mathbb{P}^4$. Then $\pi = 1 + [d(d-4)/4]$ and X is either the complete intersection of V^2 with another hypersurface, or X is linked to a plane in the complete intersection of V^2 with another hypersurface. \square*

Theorem 4.2. [Ko], [Au]. *Let X be contained in an irreducible cubic hypersurface $V^3 \subset \mathbb{P}^4$. Then either X is projectively Cohen-Macaulay and linked on V^3 to an irreducible scheme of degree ≤ 3 , or X is linked on V^3 to a Veronese surface, or to a quintic elliptic scroll. \square*

Corollary 4.3. *If X is contained in a cubic hypersurface and if $d \geq 9$, then X is of general type. \square*

To derive a lower bound for π and bounds for χ we may use Severi's Theorem [Se] together with Riemann-Roch, the Hodge index theorem, the Enriques-Kodaira classification and adjunction theory. In the context of section 3 we note:

Theorem 4.4. [Au], [La]. *If X is a scroll, then X is a rational cubic or an elliptic quintic scroll. \square*

Theorem 4.5. [BR], [ES]. *If X is a conic bundle, then X is a Del Pezzo surface of degree 4, or a Castelnuovo surface. \square*

Corollary 4.6. *If $d \geq 6$, then the adjunction map Φ is defined and $(K + H)^2 > 0$, i.e., $\dim \Phi(X) = 2$. \square*

Once the numerical invariants are fixed, we use the information on the dimensions $h^i \mathcal{J}_X(m)$ provided by Riemann-Roch and [DES, 1.7]. In some cases more information on the dimensions and the structures of the Hartshorne-Rao modules may be obtained by studying the relations between the multiseccants to X , the plane curves on X and the syzygies of \mathcal{J}_X (compare [PR]). This information is helpful for construction and classification purposes. In other cases one has to go through the adjunction process to analyze, how a given X fits into the Enriques-Kodaira classification. In any case it is crucial to know the number of exceptional lines on X . Le Barz' 6-secant formula [LB] tells us, that the number of 6-secants to X (if finite) plus the number of exceptional lines equals a polynomial expression $N_6 = N_6(d, \pi, \chi)$ (if X does not contain a line with self-intersection ≥ 0). This fits well with the ideas of section 1. Once having constructed X explicitly, we can compute the 6-secants easily. For examples we refer to [DES],[Po].

With the following example we would like to demonstrate, that the construction via syzygies is not always as straightforward as in Example 1.6.

Example 4.9. [Po] Let us construct a family of smooth surfaces $X \subset \mathbb{P}^4$ with $d = 11$, $\pi = 11$ and $\chi = 3$. In view of [DES, 1.7] a plausible Beilinson cohomology table for $\mathcal{J}_X(4)$ is

2				
	1			
		1	4	3

Suppose that a smooth surface X with these data exists. Then Beilinson's theorem yields a resolution of type

$$0 \rightarrow \mathcal{F} = 2\mathcal{O}(-1) \oplus \Omega^3(3) \xrightarrow{\varphi} \mathcal{G} \rightarrow \mathcal{J}_X(4) \rightarrow 0,$$

where \mathcal{G} is the cohomology of a monad

$$0 \rightarrow \Omega^2(2) \xrightarrow{d_1^{-2,1}} 4\Omega^1(1) \xrightarrow{d_1^{-1,1}} 3\mathcal{O} \rightarrow 0 .$$

Arguing as in example 1.6, we conversely choose $\mathcal{G} = \text{Sy}_{z_1}(M)$, where M is a module with Hilbert function $(1, 4, 3)$ and a minimal free presentation of type

$$0 \leftarrow M \leftarrow S(2) \xleftarrow{(\alpha, \beta)} S(1) \oplus 7S .$$

So M is the tensor product of the Koszul complex given by the linear form α and the module M' presented by β . We may assume that $\alpha = x_4$ and that M' is a module over $R' = \mathbb{C}[x_0, \dots, x_3]$. M' has the same Hilbert function as M , namely $(1, 4, 3)$. The general such M' has syzygies of type

$$\begin{array}{ccccccc} 0 \leftarrow M' \leftarrow R'(2) & & & & & & \\ & \swarrow \beta & & & & & \\ & & 7R' & & 8R'(-1) & & aR'(-2) \\ & & \swarrow & & \oplus & \longleftarrow & \oplus \\ & & & & (3+a)R'(-2) & & 8R'(-3) \xleftarrow{t\gamma} 3R'(-4) \leftarrow 0 , \end{array}$$

with $a = 0$. It is easy to see, that in this case no morphism $\varphi \in \text{Hom}(\mathcal{F}, \mathcal{G})$ is injective. The trick for the construction of X is to choose β special in order to obtain some extra syzygies and thus a larger space $\text{Hom}(\mathcal{F}, \mathcal{G})$. We will construct a module M' with the above type of syzygies and $a = 1$. Equivalently, we will construct the \mathbb{C} -dual module M'^* by defining its presentation matrix $\gamma = (\gamma_1, \gamma_2)$. Choose four general lines L_1, \dots, L_4 in the hyperplane $V(x_4)$, denote by ϵ the presentation matrix in the direct sum of the four Koszul complexes built on these lines and let δ be a general 3×4 -matrix with entries in \mathbb{C} . Then ϵ and thus also $\gamma_1 = \delta\epsilon$ has four linear 1-syzygies. Let γ_2 be given by 3 general quadrics. Then γ presents an artinian module as desired. With these choices the generic $\varphi \in \text{Hom}(\mathcal{F}, \mathcal{G})$ yields a smooth surface X cut out by 8 quintics and 4 sextics. In general it is a plausible guess and in many cases true that the number of 6-secants to a surface in \mathbb{P}^4 is precisely the number of sextic generators of its homogeneous ideal. Indeed, in our case it is easy to see that L_1, \dots, L_4 are precisely the 6-secants to X [Po, Proposition 3.32]. Le Barz' 6-secant formula gives $N_6(11, 11, 3) = 5$. Hence there is one exceptional line on X . One can show that there are no other exceptional curves [Po, Proposition 3.31]. Since $K^2 = 1$ by the double point formula X is of general type. \square

In some cases it is quite subtle to construct artinian modules with the desired graded Betti numbers. From this point of view the most difficult surfaces are the abelian and bielliptic surfaces known so far [ADHPR2]. These are also the surfaces with the most beautiful geometry behind (compare [ADHPR1] and Hulek' s article on the Horrocks-Mumford bundle in this volume). The link between the geometry and the syzygies is provided by the distribution of the 2- and 3-torsion points on the Heisenberg invariant elliptic normal curves in \mathbb{P}^4 . In turn, these curves are related to the Horrocks-Mumford bundle. Our knowledge on this bundle has influenced the construction of further families of surfaces (compare [ADHPR1, Thm. 32], [DES, 2.5], [Po, 4.1 and 7.4]).

5. 3-FOLDS IN \mathbb{P}^5

In this section X will denote a smooth, non-degenerate 3-fold in \mathbb{P}^5 , S a general hyperplane section, $d = H^3$ its degree and $\pi = \frac{1}{2}H^2 \cdot (K + 2H) + 1$ its sectional genus.

We have two double point formulae, one for X ,

$$K^3 = -5d^2 + d(2\pi + 25) + 24(\pi - 1) - 36\chi(\mathcal{O}_S) - 24\chi(\mathcal{O}_X),$$

and one for S , which may be rewritten as

$$H \cdot K^2 = \frac{1}{2}d(d + 1) - 9(\pi - 1) + 6\chi(\mathcal{O}_S)$$

(compare e.g. [Ok2]). So the basic invariants of X are $d, \pi, \chi(\mathcal{O}_X)$ and $\chi(\mathcal{O}_S)$. Equivalently one may consider the pluridegrees

$$d_i = (K + H)^i \cdot H^{3-i} = c_{2+i}(\mathcal{J}_X(5)), \quad i = 0, \dots, 3,$$

introduced in [BBS]. By Zak's theorem [Za] X is linearly normal. Moreover $h^1(X, \mathcal{O}_X) = 0$ by Barth-Larsen-Lefschetz [BL]. In particular S is linearly normal and regular. Clearly X is projectively Cohen-Macaulay iff S has this property. So by studying S we obtain from Theorem 4.1 and Theorem 4.2:

Proposition 5.1. *Let X be contained in a cubic hypersurface. Then X is projectively Cohen-Macaulay. In particular X is of general type if $d \geq 13$. \square*

To work out a finite list of admissible invariants for a given degree one may again start with Halphen's upper bound for π . Further tools are a congruence obtained from Riemann-Roch [BSS2, 0.11], the inequalities deduced from the semipositivity of $\mathcal{N}_{X/\mathbb{P}^5}(-1)$ [BOSS1, Proposition 2.2] and adjunction theory. In the context of section 3 we recall some classification results. X_1, \dots, X_{30} will denote the 3-folds listed in table 7.3. The first result follows from Theorem 4.4 and Theorem 4.5 (compare also [BOSS2]).

Proposition 5.2.

- (i) *If X is a scroll over a smooth curve, then $X = X_1$ is a Segre cubic scroll.*
- (ii) *If X is a Fano 3-fold of index 2, then $X = X_2$ is a complete intersection of two quadric hypersurfaces.*
- (iii) *If X is a quadric bundle over a smooth curve, then $X = X_3$ is a Castelnuovo 3-fold. \square*

Theorem 5.3. [Ott]. *If X is a scroll over a smooth surface, then X is one of the following:*

- (i) *a Segre scroll $X = X_1$,*
- (ii) *a Bordiga scroll $X = X_4$,*
- (iii) *a Palatini scroll $X = X_6$,*
- (iv) *a scroll $X = X_{11}$ over a K3 surface. \square*

From now on we suppose that X is none of the exceptional 3-folds above. Then the adjunction map Φ is defined and the connected morphism r of its Stein factorization contracts the linear $\mathbb{P}^2 \subset X$ with normal bundle $\mathcal{O}_{\mathbb{P}^2}(-1)$ to points.

Proposition 5.4. [BSS2] r is an isomorphism unless $X = X_7$. \square

From now on we suppose that $X \neq X_7$. Then X coincides with its first reduction.

The next step in adjunction theory is to study $K + H$. This is big and nef unless X is one of the special varieties listed in [So5]. In our case these are classified:

Theorem 5.5. [BOSS2]. $K + H$ is big and nef unless

- (i) (X, H) is a Fano 3-fold of index 1. Then $X = X_5$ is a complete intersection of type $(2, 3)$.
- (ii) (X, H) is a Del Pezzo fibration over a smooth curve. Then $X = X_8$ or $X = X_9$.
- (iii) (X, H) is a conic bundle over a surface. Then $X = X_{12}$ or $X = X_{20}$. \square

From now on we suppose that $K + H$ is big and nef. Then S is of general type and minimal. Therefore X is called to be of *log-general type* [BSS1]. In this case further numerical information is provided by the generalized Hodge index theorem [BBS, Lemma 1.1] and the parity relations [BBS, Lemma 1.4].

From the Kawamata-Shokurov base point free theorem (see [KMM, §3]) we know that for some $m > 0$ the linear system $|m(K + H)|$ gives rise to a morphism, say $\Psi : X \rightarrow X''$. For m large enough we can assume that Ψ has connected fibers and normal image. We write $L'' = \Psi_*(H)$, $K'' = K_{X''}$ and $H'' = K'' + L''$. Then L'' and H'' are ample and

$$\Psi^*(H'') \equiv K + H$$

(cf. [BFS, (0.2.6)]). (X'', L'') is called the *second reduction* of (X, H) [So4],[BFS].

Proposition 5.6. [BSS3, Corollary 1.3] If $d \neq 10$ and $d \neq 13$, then X'' is smooth and Ψ is an isomorphism outside a disjoint union \mathcal{C} of smooth curves. Let C be an irreducible component of \mathcal{C} and let $D := \Psi^{-1}(C)$. Then the restriction Ψ_D of Ψ to D is a \mathbb{P}^1 -bundle $\Psi_D : D \rightarrow C$ and $\mathcal{N}_{D|F}^X \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ for any fiber F of Ψ_D . In fact, Ψ is simply the blowing up along \mathcal{C} . \square

Remark 5.7. i) If $d = 10$, then there is exactly one case where X'' is not smooth. Namely, for $X = X_{16}$ the second reduction morphism Ψ , which is defined by $|K + H|$, contracts the quadric surface K to a singular point p . Moreover, $\Psi(X) \subset \mathbb{P}^6$ is a complete intersection of type $(2, 2, 3)$, while X is the projection from p of $\Psi(X)$ (see also section 7).
 ii) From [BSS3, (0.5.1) and (1.1)] and [Ed, (3.1.3)] it follows that in case $d = 13$ the second reduction is singular iff there exist on X divisors $D \cong \mathbb{P}^2$, with $\mathcal{N}_D^X \cong \mathcal{O}_{\mathbb{P}^2}(-2)$, which are contracted to points. We are not aware of any such example. \square

Example 5.8. Let $X \subset \mathbb{P}^5$ be a smooth 3-fold with $d = 11$ and $\pi = 14$. Then $\chi(\mathcal{O}_S) = 8$ and $\chi(\mathcal{O}_X) = 0$ (compare [BSS2]). Every smooth surface in \mathbb{P}^4 with the same invariants as S is linked $(4, 4)$ to a Castelnuovo surface [Po, Prop. 3.70]. In particular S and hence X are projectively Cohen-Macaulay with syzygies of type

$$0 \rightarrow 2\mathcal{O}(-5) \oplus \mathcal{O}(-6) \xrightarrow{\varphi} 4\mathcal{O}(-4) \rightarrow \mathcal{J}_X \rightarrow 0 .$$

Consequently $X = X_{18}$ is linked $(4, 4)$ to a Castelnuovo 3-fold. Conversely this shows the existence of 3-folds of type X_{18} [BSS2].

What kind of 3-fold is X ?

From the invariants we compute the Kodaira dimension $\kappa(X) = 0$. In order to show that X is a blown up Calabi-Yau 3-fold we study $|K + H|$. By dualizing φ we obtain the resolution

$$0 \leftarrow \omega_X(1) \leftarrow \mathcal{O}(1) \oplus 2\mathcal{O} \xleftarrow{\varphi} 4\mathcal{O}(-1) \leftarrow \mathcal{O}(-5) \leftarrow 0 .$$

So $|K + H|$ is base point free, $\dim |K + H| = 7$, and we have a well-defined map $\Psi_{|K+H|} : X \rightarrow X''$, where X'' is a 3-fold in \mathbb{P}^7 . Moreover $h^0(\mathcal{O}_S(K_S - H_S)) = h^2(\mathcal{O}_S(1)) = 1$ by Riemann-Roch and Severi's theorem, thus S is minimal and there exists a rigid curve $D \in |K_S - H_S|$ with $H_S \cdot D = 4$, $p_a(D) = 0$. In particular, $|K_S| = |D + H_S|$ defines an embedding outside the support of D and maps the divisor D onto a line L in $\mathbb{P}^6 = \mathbb{P}(H^0(\mathcal{O}_S(K_S)))$. It follows that $\Psi = \Psi_{|K+H|} : X \rightarrow X'' \subset \mathbb{P}^7$ coincides with the second reduction morphism. Moreover, by Proposition 5.6, X'' is smooth, K is a smooth rational scroll $\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{(1,2)} \mathbb{P}^5$, which is contracted by Ψ to the line $L \subset X''$, while X is exactly the blow up of X'' along this line. Riemann-Roch gives $\chi(\mathcal{O}_X(2H + 2K)) = 32$, hence $h^0(\mathcal{J}_{X''}(2)) \geq h^0(\mathcal{O}_{\mathbb{P}^7}(2)) - h^0(\mathcal{O}_X(2H + 2K)) = 4$. In other words, X'' lies on 4 linearly independent hyperquadrics. In fact, as one can check, $X'' \subset \mathbb{P}^7$ is the complete intersection $\Sigma_{(2,2,2,2)}$ of 4 hyperquadrics. Conversely, let L be a line in \mathbb{P}^7 and $\Sigma_{(2,2,2,2)} \subset \mathbb{P}^7$ a smooth complete intersection of 4 hyperquadrics containing L . Then a general projection $X = \text{proj}_L \Sigma_{(2,2,2,2)} \subset \mathbb{P}^5$ is a 3-fold of type X_{18} . \square

Remark 5.9. Similarly, [Po, Proposition 3.59] yields an easy proof for the uniqueness of the examples of smooth 3-folds with $d = 11$ and $\pi = 13$ constructed in [BSS2]. The uniqueness for the other two families with $d = 11$ in [BSS2] is clear from [GP]. \square

The construction via syzygies of all smooth 3-folds $X \subset \mathbb{P}^5$ known so far is straightforward. Nevertheless, it is sometimes quite subtle to determine the structure of X . We will give examples of this kind in the next section.

6. EXAMPLES: TWO FAMILIES OF BIRATIONAL CALABI-YAU 3-FOLDS IN \mathbb{P}^5

In this section we will construct and study a family of smooth 3-folds $X \subset \mathbb{P}^5$ with $d = 17$, $\pi = 32$, $\chi(\mathcal{O}_X) = 0$ and $\chi(\mathcal{O}_S) = 24$. We will also describe a family of smooth 3-folds $X' \subset \mathbb{P}^5$ obtained via linkage $X' \underset{(5,6)}{\sim} X$.

Let us first explain how to construct X via syzygies. In view of Riemann-Roch the following is a plausible Beilinson cohomology table for $\mathcal{J}_X(5)$:

1					
	4	2			
					2

Suppose that a 3-fold X with these data exists. Then Beilinson's theorem yields an exact sequence

$$0 \rightarrow \mathcal{F} = \mathcal{O}(-1) \oplus 4\Omega^4(4) \xrightarrow{\varphi} \mathcal{G} = 2\Omega^3(3) \oplus 2\mathcal{O} \rightarrow \mathcal{J}_X(5) \rightarrow 0. \quad (6.1)$$

Conversely, as one can check, the minors of a generic $\varphi \in \text{Hom}(\mathcal{F}, \mathcal{G})$ vanish along a smooth 3-fold X as desired. By construction, \mathcal{J}_X has syzygies of type

$$0 \leftarrow \mathcal{J}_X \leftarrow 2\mathcal{O}(-5) \oplus 5\mathcal{O}(-6) \leftarrow 8\mathcal{O}(-7) \leftarrow 2\mathcal{O}(-8) \leftarrow 0. \quad (6.2)$$

What kind of 3-fold is X ? From the syzygies we see that X can be linked (5, 5) to a 3-fold Z of degree 8. It is not too hard to identify the scheme Z .

Starting conversely with Z we will reconstruct X and study its geometry. Z can be described as follows: Let $Y = \mathbb{P}^1 \times \mathbb{P}^2 \xrightarrow{(1,1)} \mathbb{P}^5$ be a Segre cubic scroll and let L_1, \dots, L_5 be five general lines in \mathbb{P}^2 . Then for $i = 1, \dots, 5$ the quadric $Q_i = \mathbb{P}^1 \times L_i \xrightarrow{(1,1)} \mathbb{P}^5$ is contained in Y and spans a linear subspace $\Pi_i \subset \mathbb{P}^5$ of dimension 3. Clearly, $\Pi_i \cap Y = Q_i$ (Y is cut out by quadrics) and $\Pi_i \cap \Pi_j = \mathbb{P}^1 \times \{p_{ij}\}$, where $\{p_{ij}\} = L_i \cap L_j$ for $i < j$. Hence the scheme

$$Z := Y \cup \Pi_1 \cup \dots \cup \Pi_5$$

is locally Cohen-Macaulay, and moreover a local complete intersection outside the lines $L_{ij} := \mathbb{P}^1 \times \{p_{ij}\}$. Write $Z_k = Y \cup \bigcup_{i=1}^k \Pi_i$, $k = 0, \dots, 5$. Then $Z_0 = Y$ and $Z_5 = Z$. From the exact sequences

$$0 \rightarrow \mathcal{J}_{Z_{k-1}}(m-1) \rightarrow \mathcal{J}_{Z_k}(m) \rightarrow \mathcal{J}_{Z_{k-1} \cap \mathbb{P}^4, \mathbb{P}^4}(m-1) \rightarrow 0, \quad (6.3)$$

where $\mathbb{P}^4 \subset \mathbb{P}^5$ is a general hyperplane through Π_k , we deduce that $h^0 \mathcal{J}_Z(3) = 0$, $h^0 \mathcal{J}_Z(4) = 1$ and $h^0 \mathcal{J}_Z(5) = 26$, and that $\mathcal{J}_Z(5)$ is globally generated.

Proposition 6.4. *Let X be linked to Z in the complete intersection of two general quintic hypersurfaces containing Z . Then X is smooth, it contains the lines L_{ij} and $\mathcal{J}_X(5)$ has a resolution of type (6.1).*

Proof. By a variant of Theorem 2.1 (compare [PS]) X is smooth outside the lines L_{ij} . By using the exact sequences (6.3) we see that the general quintic hypersurface through Z contains the first infinitesimal neighborhood of L_{ij} , which is a multiplicity 5 structure on such a line. Higher infinitesimal neighborhoods are not contained in the general quintic hypersurface through Z . Moreover the tangent cone to Z at a point $p \in L_{ij}$ is $\Pi_i \cup \Pi_j \cup \mathbb{T}_p Y$, and $\Pi_i \cap \mathbb{T}_p Y = \mathbb{T}_p Q_i$. Now a local computation shows that indeed X is smooth along and contains the lines L_{ij} . That $\mathcal{J}_X(5)$ has a Beilinson cohomology table as above follows via liaison from the exact sequences (6.3). \square

Remark 6.5. (i) By dualizing (6.1) we find that $\omega_X(1)$ has a presentation of type

$$0 \leftarrow \omega_X(1) \leftarrow \mathcal{O}(1) \oplus 18\mathcal{O} \leftarrow 50\mathcal{O}(-1) \leftarrow \dots$$

Thus $|K + H|$ is base point free and $\dim |K + H| = 24$.

(ii) From the double point formulae we compute

$$H^2 \cdot K = 28, \quad H \cdot K^2 = 18 \quad \text{and} \quad K^3 = -52 .$$

In particular $(K + H) \cdot K^2 = -34$, hence $\kappa(X) \leq 1$. In fact, as we will see later, X is a birational Calabi-Yau 3-fold. \square

We use in the sequel the above liaison to describe the geometry of X :

Lemma 6.6. *Each linear subspace Π_i intersects X along a smooth sextic surface S_i . A general element in the residual pencil $|H - S_i|$ is a smooth blown-up K3 surface of degree 11, sectional genus 12, which is embedded in its corresponding \mathbb{P}^4 by a linear system of type*

$$|H_{min} - 2E_1 - \sum_{i=2}^{10} E_i|, \quad \text{with} \quad H_{min}^2 = 24 .$$

Proof. It follows from the standard liaison exact sequences that X meets Π_i along a divisor in the class $4H_{\Pi_i} - K_{\Pi_i} - Q_i$, hence along a sextic surface S_i , which is smooth for general choices in the liaison. For the second statement in the lemma, we observe that a general element in $|H - S_i|$ is linked (4, 4) inside the hyperplane H to the configuration of planes $P_i \cup \bigcup_{j \neq i} (H \cap \Pi_j)$, where P_i is the plane residual to Q_i in the intersection $H \cap Y$. Therefore the lemma follows from the following:

Proposition 6.7. [Po] *Let T be a configuration $T = P \cup P_1 \cup P_2 \cup P_3 \cup P_4 \subset \mathbb{P}^4$, where P is a plane, while P_i , $i = 1, \dots, 4$, is a plane meeting P along a line, such that no three of the lines have common intersection points. Then T can be linked in the complete intersection of two general quartic hypersurfaces to a smooth, non-minimal K3 surface $S \subset \mathbb{P}^4$ with $d = 11$ and $\pi = 12$, embedded by a linear system*

$$H_S \equiv H_{min} - 2E_1 - \sum_{i=2}^{10} E_i, \quad H_{min}^2 = 24 .$$

Moreover, P meets S along the exceptional conic E_1 and an extra scheme of length 6, while each intersection $P_i \cap S$ is a plane quintic curve. Residual to it there is a base point free pencil of elliptic space curves of degree 6. \square

Lemma 6.8. $|H - S_i|$ is a base point free pencil, $i = 1, \dots, 5$.

Proof. Let H_i denote a general hyperplane containing Π_i , and let K_i be the surface residual to S_i in $H_i \cap X$. Then $S_i \cap K_i \subset \Pi_i \cap K_i$, and in fact equality holds since $\deg S_i \cap K_i = 32 - \pi(S_i) - \pi(K_i) + 1 = 11$. Thus if $C_i = S_i \cap K_i$, then $C_i^2 = 2p_a(C_i) - 2 - K_{S_i} \cdot C_i = 0$, where the intersection numbers are computed on S_i , and the lemma follows. \square

As a corollary, we deduce that X is an elliptic 3-fold, namely

Corollary 6.9. *For all $i \neq j$, the linear system $|H - S_i| \boxtimes |H - S_j|$ induces an elliptic fibration*

$$\varphi_{|H-S_i| \boxtimes |H-S_j|} : X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1,$$

with elliptic space curves of degree 6 as fibres.

Proof. Fix a general point in $\mathbb{P}^1 \times \mathbb{P}^1$, i.e., two general hyperplanes, H_i containing Π_i and H_j containing Π_j , and denote as above by K_i and K_j the residual surfaces to S_i and S_j respectively. By Proposition 6.7, $H_i \cap \Pi_j \cap K_i$ and $H_j \cap \Pi_i \cap K_j$ are plane quintic curves, hence $K_i \cap K_j$ is an elliptic space curve of degree 6, namely the residual to $H_i \cap \Pi_j \cap K_i$ in $H_j \cap K_i$, or equivalently the residual to $H_j \cap \Pi_i \cap K_j$ in $H_i \cap K_j$. \square

By liaison we deduce that X meets the Segre scroll Y along a surface T_2 in the class $4H_Y - K_Y - \sum_{i=1}^5 Q_i$, thus along a conic bundle of degree 10 and sectional genus 6. Moreover, the standard liaison exact sequences yield on X the linear equivalence

$$4H - K \equiv T_2 + \sum_{i=1}^5 S_i. \quad (6.10)$$

We study in the sequel the structure of the map defined by the composition of the cartesian product of the 5 pencils $|H - S_i|$, $i = 1, \dots, 5$, with the Segre embedding to \mathbb{P}^{31} :

$$\Upsilon = \Upsilon_{|5H - \sum_{i=1}^5 S_i|} : X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^{31}. \quad (6.11)$$

Lemma 6.12. *The canonical divisor K of X has two components T_1 and T_2 . T_1 is a scroll of degree 18 and sectional genus 10, while T_2 is the above conic bundle of degree 10 and sectional genus 6.*

Proof. Let, as in the proof of Lemma 6.8, K_i be a general element in the pencil $|H - S_i|$. We recall that $(H - S_i)^2 = 0$, thus $K|_{K_i} \equiv K + (H - S_i)|_{K_i} \equiv K_{H-S_i}$. In other words, K meets a $K3$ surface K_i along its canonical divisor, namely, by Proposition 6.7, along 9 exceptional lines and one exceptional conic in the plane residual to Q_i in $H \cap Y$. In conclusion, the exceptional conics sweep the conic bundle T_2 , which is thus a component of K , while the exceptional lines on the K_i 's are rulings of a scroll T_1 of degree $H^2 \cdot K - 10 = 18$. Since $\omega_{S_i} = \mathcal{O}_{S_i}(2)$ and $S_i \cap S_j = L_{ij}$, (6.10) restricted to S_i yields $T_2 \cap S_i \equiv 2H_{S_i} - \sum_{j \neq i} L_{ij}$.

On the other side from (6.10) again we infer:

$$(H - S_i)|_{S_i} + (2H - \sum_{j \neq i} S_j)|_{S_i} + 2H|_{S_i} \equiv T_2|_{S_i} + H|_{S_i} + K|_{S_i}.$$

Thus, since $(H - S_i)|_{S_i} \equiv H_{K_i}$, it follows that K intersects S_i along a hyperplane section of X . We deduce that T_1 must intersect S_i along a curve of degree 9 and genus 10, which in turn must be a section of this scroll since it meets the exceptional lines of K_i in one point. In other words, T_1 is a scroll of degree 18 and genus 10. \square

Lemma 6.13.

- i) *The linear system $|K + H|$ defines a birational morphism $\Psi = \Psi_{|K+H|} : X \rightarrow \Psi(X) \subset \mathbb{P}^{23}$, which contracts the scroll T_1 to a curve of degree 27. Moreover, X is the blowing up of $\Psi(X)$ along this curve.*
- ii) *The morphism $\Upsilon = \Upsilon_{|5H - \sum_{i=1}^5 S_i|} : X \rightarrow \Upsilon(X) \subset \mathbb{P}^{31}$, induced by $|H + K + T_2|$, contracts the conic bundle T_2 to a curve and is birational on its image.*

Proof. Let K_i be a general element of the pencil $|H - S_i|$. Part i) follows easily since $|K + H|$ induces on K_i the adjunction morphism $\Phi_i = \Phi_{|H_{K_i} + K_{K_i}|} : K_i \rightarrow \Phi_i(K_i) \subset \mathbb{P}^{12}$, which is birational and blows down only the 9 exceptional lines $K \cap K_i$. A similar argument works for part ii) since $|5H - \sum_{i=1}^5 S_i| = |H + K + T_2|$ restricts to K_i as the map onto the second adjoint surface given by the adjunction process. \square

We can show now that X is a non minimal Calabi-Yau 3-fold, namely:

Proposition 6.14. *The morphism*

$$\Upsilon = \Upsilon_{|5H - \sum_{i=1}^5 S_i|} : X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^{31}$$

is birational on its image and contracts only the canonical divisor of X to a curve. Moreover, the image $\Upsilon(X)$ is a smooth complete intersection of type $(1, 1, 1, 1, 1)^2$ in $\times_{i=1}^5 \mathbb{P}^1$, hence a minimal Calabi-Yau 3-fold in \mathbb{P}^{29} .

Proof. The smoothness of $\Upsilon(X)$ follows from the fact that the iterated adjunction morphisms for K_i blow down only the (-1) -lines and (-1) -conics onto the minimal model of K_i . To see further that $\Upsilon(X)$ is a complete intersection of the type claimed we need to compute some intersection numbers. By Lemma 6.8, $(H - S_i)^2 = 0$, thus $(H - S_i)^2 \cdot H = 0$ and $(H - S_i)^2 \cdot S_j = 0$, which yields $H \cdot S_i^2 = -5$, $S_i^2 \cdot S_j = -4$, for $i \neq j$, and $S_i^3 = -16$. Moreover $S_i \cdot S_j \cdot S_k = 0$, for $i \neq j \neq k$, $i \neq k$, since $\Pi_i \cap \Pi_j \cap \Pi_k \subset L_{ij} \cap L_{ik} \cap L_{jk} = \emptyset$, and so we deduce that $\deg \Upsilon(X) = (5H - \sum_{i=1}^5 S_i)^3 = 120$. On the other side, $\deg \times_{i=1}^5 \mathbb{P}^1 = 5! = 120$ in \mathbb{P}^{31} , while $\Upsilon(X)$ spans only a \mathbb{P}^{29} since $h^0(\mathcal{O}_X(H + K + T_2)) \leq h^0(\mathcal{O}_X(H + 2K)) = \chi(\mathcal{O}_X(H + 2K)) = 30$. The proposition follows. \square

Proposition 6.15. *Let V^5 and V^6 be general hypersurfaces of degrees 5 and 6 resp. containing X . Then X can be linked in the complete intersection of V^5 and V^6 to a smooth 3-fold $X' \subset \mathbb{P}^5$. X' has invariants $d' = 13$, $\pi' = 18$, $\chi(\mathcal{O}_{X'}) = 0$, $\chi(\mathcal{O}_{S'}) = 10$ and $p_g(X') = h^0(\mathcal{J}_X(5)) - 1 = 1$. Hence $(H')^2 \cdot K' = 8$, $H' \cdot (K')^2 = -2$ and $(K')^3 = -4$ by the double point formulae.*

Proof. Smoothness follows from a Bertini argument since, on V^5 , X is cut out by sextic hypersurfaces (compare 2.1 and [PS]). The numerical information follows from the standard liaison exact sequences. \square

Corollary 6.16. *X' is the degeneracy locus of a morphism*

$$0 \rightarrow \mathcal{O}(-1) \oplus 2\Omega^2(2) \rightarrow 4\Omega^1(1) \oplus 2\mathcal{O} \rightarrow \mathcal{J}_{X'}(5) \rightarrow 0 .$$

Hence $\mathcal{J}_{X'}$ has syzygies of type

$$0 \leftarrow \mathcal{J}_{X'}(5) \leftarrow 2\mathcal{O} \oplus 19\mathcal{O}(-1) \leftarrow 50\mathcal{O}(-2) \leftarrow 48\mathcal{O}(-3) \leftarrow 22\mathcal{O}(-4) \leftarrow 4\mathcal{O}(-5) \leftarrow 0 .$$

Proof. This follows from (6.1) via liaison or by applying Beilinson's theorem. \square

What type of 3-fold is X' ?

Proposition 6.17. *K' is a smooth scroll of degree 8, sectional genus 3 over a plane quartic curve. Moreover, the Segre scroll $Y = \mathbb{P}^1 \times \mathbb{P}^2$ meets X' along the scroll K' and a curve of degree 9, arithmetic genus 4 on the scroll T_2 .*

Proof. From general liaison arguments it follows that Z intersects X' along the canonical divisor of X' plus may be something of bigger codimension. On the other side, $V^6 \cap \Pi_i = S_i$ since $\Pi_i \cap X = S_i$ for all i . We deduce that the 2-dimensional part of the scheme theoretical intersection $Y \cap X'$ is exactly K' . Now $\text{Pic}(Y)$ is generated by the classes $P = [\{\text{point}\} \times \mathbb{P}^2]$ and $Q = [\mathbb{P}^1 \times \mathbb{P}^1]$, and $P^2 = 0$, $Q^3 = 0$, $Q^2 \cdot P = 1$. Then the scroll T_2 is of class $4H_Y - K_Y - \sum_{i=1}^5 Q_i \equiv 6P + 2Q$. But K' is residual to T_2 in $Y \cap V^6$. Moreover, T_2 is cut out on Y outside the Π_i 's by the sextic hypersurfaces through X . It follows that K' is smooth for a general choice of the liaison, and that K' is of class $6H_Y - T_2 \equiv 4Q$. In particular, K' is a scroll over a plane quartic curve and has the claimed invariants. Outside K' , X' can meet the scroll Y only inside $T_2 \subset Y \cap X$. The proposition follows now since $X' \cap T_2 \equiv (5H_X - K_X) \cdot T_2$ is a curve of degree 41, arithmetic genus 80, with a component of degree 32 on the scroll K' . \square

Proposition 6.18.

- i) $|H' + K'|$ is base point free and big.
- ii) $\Psi' = \Psi_{|H'+K'|} : X' \rightarrow \mathbb{P}^9$ is birational on its image $M = \Psi'(X')$, which is a smooth Calabi-Yau 3-fold, with $\deg M = 27$, $\pi(M) = 28$ and $c_3(M) = -64$.
- iii) Ψ' contracts the scroll K' to a curve of degree 6 and genus 3, and is an isomorphism outside this scroll. Moreover, X' is the blow up of M along this curve.

Proof. i) From the syzygies we see that $\omega_{X'}$ is a quotient of $\mathcal{O} \oplus 4\mathcal{O}(-1)$, thus $|H' + K'|$ is base point free and big since $(H' + K')^3 = 27$. Moreover $\dim |H' + K'| = 9$

ii) From the liaison exact sequence

$$0 \rightarrow \mathcal{J}_{V^5 \cap V^6}(6) \rightarrow \mathcal{J}_X(6) \rightarrow \omega_{X'}(1) \rightarrow 0$$

we deduce that the map $\Psi' : X' \rightarrow \mathbb{P}^9$ is in fact the composition of the restriction to X' of the rational morphism $\Xi : \mathbb{P}^5 \dashrightarrow \mathbb{P}^{16} = \mathbb{P}(H^0(\mathcal{J}_X(6)))$ given by the sextic hypersurfaces through X , with a projection from $\mathbb{P}^{16} \dashrightarrow \mathbb{P}^9$ along $\mathbb{P}^6 = \mathbb{P}(H^0(\mathcal{J}_{V^5 \cap V^6}(6)))$. Thus in order to show that Ψ' is birational on its image it is enough to check that Ξ is birational on its image and that the projection $\mathbb{P}^{16} \dashrightarrow \mathbb{P}^9$ is generic. But one checks easily that 5 general sextic hypersurfaces through X meet in exactly one point outside X . In particular, it follows that $\Psi' : X' \rightarrow M$ coincides with the second reduction map of X' .

iii) Since $(H' + K') \cdot K' = 0$, Ψ' contracts the scroll K' . Its image is isomorphic to the plane quartic curve, which is the base of the scroll K' . From Remark 5.7 it follows that M is smooth and Ψ' is an isomorphism outside the scroll K' , unless there are divisors $D \cong \mathbb{P}^2 \subset X'$ with $N_D^{X'} \cong \mathcal{O}_{\mathbb{P}^2}(-2)$, which are contracted to singular points on M . Assume that such a divisor D exists. Then a general hyperplane section S' of X' contains a (-2) -line L . But on S' , $h^0(\mathcal{O}_{S'}(K_{S'} - H_{S'})) = h^3(\mathcal{O}_{X'}) = 1$, so if $D_{S'} \in |K_{S'} - H_{S'}|$, then $K_{S'} \cdot L = 0 = 1 + D_{S'} \cdot L$, and thus L must be a component of $D_{S'}$. On the other side

$D_{S'}$ is an irreducible hyperplane section of the smooth scroll $K_{X'}$, and therefore contains no such line as a component. \square

7. OVERVIEW

In this section we collect some information on the families of smooth non general type surfaces in \mathbb{P}^4 and 3-folds in \mathbb{P}^5 known to us.

 Table 7.1. Known families of smooth non general type surfaces in \mathbb{P}^4

degree	Enriques-Kodaira Classification						
	rational	ruled irrat.	Enriques	$K3$	abelian	bielliptic	elliptic
$d \leq 4$	6			1			
$d = 5$	1 [Ca]	1 [Seg]					
$d = 6$ [Io1],[Ok1]	1 [Bo],[Ve] [Wh]			1			
$d = 7$ [Io1],[Ok3]	1 [Io1],[Ok3]			1 [Ro1]			1 [Ba]
$d = 8$ [Io2],[Ok4]	2 [Ok4],[A11]			1 [Ok4]			1 [Ba]
$d = 9$ [AR]	2 [A11],[A12]		1 [Cos],[CV]	1 [Ro1]			1 [AR]
$d = 10$ [Ra],[PR]	3 [DES],[Ra]		1 [DES],[Br]	2 [Ra],[Po]	1 [Co],[HM] [HL],[Ram]	1 [Ser] [ADHPR1]	2 [Ra]
$d = 11$ [Po]	3+2 [DES],[Po]		1 [DES]	5 [DES],[Po]			1 [Po]
$d = 12$				1 [DES]			3 [Po]
$d = 13$			2 [DES],[Po]	1 [Po]			
$d = 14$				1 [Po]			
$d = 15$					2 [HM],[Po]	1 [ADHPR1]	

Remark 7.2. (i) The classification of smooth surfaces in \mathbb{P}^4 is complete up to degree 10, and there is a partial classification in degree 11. In the first column of Table 7.1 we refer to the papers, where one can find the classification results. In the other columns we indicate the number of families known and the corresponding references. The classification up to degree 5 is classical. More information can be found in [DES, Appendix B], [Po, Appendix] and [ADHPR2].

(ii) Two families of rational surfaces of degree 11 are due to Schreyer (unpublished).

(iii) One of the families of K3 surfaces of degree 11 has been first constructed by Ranestad (compare [Po, Proposition 3.41]). \square

Table 7.3. Known families of smooth, non-degenerate, non general type 3-folds in \mathbb{P}^5

X	d	π	p_g	$\chi(\mathcal{O}_X)$	$\chi(\mathcal{O}_S)$	$\kappa(X)$	liaison	type	classification	ref.
X_1	3	0	0	1	1	$-\infty$	$X_1 \stackrel{(2,2)}{\sim} \mathbb{P}^3$	Segre embedd. of $\mathbb{P}^1 \times \mathbb{P}^2$	rational scroll	
X_2	4	1	0	1	1	$-\infty$	$X_2 = \Sigma_{(2,2)}$	Fano 3-fold of index 2	rational	[K1]
X_3	5	2	0	1	1	$-\infty$	$X_3 \stackrel{(2,3)}{\sim} \mathbb{P}^3$	Castelnuovo 3-fold; quadric fibration over \mathbb{P}^1 via $ K + 2H $	rational	[Io1],[Ok2]
X_4	6	3	0	1	1	$-\infty$	$X_4 \stackrel{(3,3)}{\sim} X_1$	Bordiga 3-fold; $X_4 = \mathbb{P}(\mathcal{E})$, \mathcal{E} rk 2 vb on \mathbb{P}^2 $c_1 = 4$, $c_2 = 10$, via $ K + 2H $	rational scroll	[Io1],[Ok2]
X_5	6	4	0	1	2	$-\infty$	$X_5 = \Sigma_{(2,3)}$	Fano 3-fold of index 1	unirational not rational	[Io1],[Ok2] [En],[Fa1]
X_6	7	4	0	1	1	$-\infty$	X_6 proj. Buchsbaum	$X_6 = \mathbb{P}(\mathcal{E})$, \mathcal{E} rk 2 vb on $\mathbb{P}^2(x_1, \dots, x_6)$; via $ K + 2H $	rational scroll	[Io1],[Ok3] [Pa]
X_7	7	5	0	1	2	$-\infty$	$X_7 \stackrel{(3,3)}{\sim} \Sigma_{(1,2)}$	$X_7 = \Sigma_{(2,2,2)}(x_0)$, the blowing up of a c.i. $\Sigma_{(2,2,2)} \subset \mathbb{P}^6$; via $ K + 2H $	blown up Fano of index 1	[Io1],[Ok3]
X_8	7	6	0	1	3	$-\infty$	$X_8 \stackrel{(2,4)}{\sim} \mathbb{P}^3$	Del Pezzo fibration over \mathbb{P}^1 , with gen. fibre $\mathbb{P}^2(x_1, \dots, x_6)$; via $ K + H $	rational	[Io1],[Ok3]
X_9	8	7	0	1	3	$-\infty$	$X_9 \stackrel{(3,3)}{\sim} \mathbb{P}^3$	Del Pezzo fibration over \mathbb{P}^1 ; gen. fibre c.i. $(2, 2)$ in \mathbb{P}^4 ; via $ K + H $	rational	[Io2]
X_{10}	8	9	1	0	6	0	$X_{10} = \Sigma_{(2,4)}$	minimal Calabi-Yau 3-fold		
X_{11}	9	8	0	2	2	$-\infty$	X_{11} proj. Buchsbaum	\mathbb{P}^1 bundle over a minimal $K3$ surface $S \subset \mathbb{P}^8$; via $ K + H $	scroll, not rational	[Ch3]
X_{12}	9	9	0	1	4	$-\infty$	$X_{12} \stackrel{(3,4)}{\sim} X_3$	conic bundle over \mathbb{P}^2 , via $ K + H $	rational	[BSS1]
X_{13}	9	10	1	0	6	0	$X_{13} = \Sigma_{(3,3)}$	minimal Calabi-Yau 3-fold		
X_{14}	9	12	2	-1	9	1	$X_{14} \stackrel{(2,5)}{\sim} \mathbb{P}^3$	minimal $K3$ fibration over \mathbb{P}^1 , via $ K $; $4K + 3H > 0$. log-gen type		[BSS1]
X_{15}	10	11	0	1	5	$-\infty$	$X_{15} \stackrel{(4,4)}{\sim} X_4$	log-general type; $ K + H $ is birational onto \mathbb{P}^3	rational	[BSS1]
X_{16}	10	12	1	0	7	0	$X_{16} \stackrel{(3,4)}{\sim} \Sigma_{(1,2)}$	$X_{16} = \text{proj}_p \Sigma_{(2,2,3)}$, with $\Sigma_{(2,2,3)} \subset \mathbb{P}^6$ c.i. singular at p ; via the inverse of $ K + H $	birational Calabi-Yau; $H^2K = 2$	[BSS1]
X_{17}	11	13	0	1	6	$-\infty$	$X_{17} \stackrel{(4,5)}{\sim} X_{11}$	blown up Fano 3-fold; $ K + H $ is birational onto a hypercubic in \mathbb{P}^4 ;	unirational not rational	[Ch3] [BSS2]
X_{18}	11	14	1	0	8	0	$X_{18} \stackrel{(4,4)}{\sim} X_3$	$X_{18} = \text{proj}_L \Sigma_{(2,2,2,2)}$, with $L \subset \Sigma_{(2,2,2,2)} \subset \mathbb{P}^7$ smooth c.i. and L a line; via $ K + H $	birational Calabi-Yau; $H^2K = 4$	[BSS2]
X_{19}	11	15	2	-1	10	1	$X_{19} \stackrel{(3,4)}{\sim} \mathbb{P}^3$	minimal $K3$ fibration over \mathbb{P}^1 via $ K $; fibres are $(2, 3)$ c.i. in \mathbb{P}^4		[BSS2]

X	d	π	p_g	$\chi(\mathcal{O}_X)$	$\chi(\mathcal{O}_S)$	$\kappa(X)$	liaison	type	classification	ref
X_{20}	12	15	0	2	6	$-\infty$	$X_{20} \stackrel{(5,5)}{\sim} X_{25}$	conic bundle over a $K3$ quartic surface $S \subset \mathbb{P}^3$, via $ K + H $	not rational	[BOSS2]
X_{21}	12	15	0	1	7	$-\infty$	$X_{21} \stackrel{(5,5)}{\sim} X_{15} \cup X_1$	$ K + H $ defines a birational map onto a Bordiga $X_4 \subset \mathbb{P}^5$	rational	[Ed]
X_{22}	12	16	1	0	9	0	$X_{22} \stackrel{(5,5)}{\sim} X_{27}$	$ K + H $ defines a birational morphism onto $\mathbb{P}(R_2) \cap Bl_{\mathbb{P}^2} \mathbb{P}^8$; see [Ch3] for details	birational Calabi-Yau; $H^2K = 6$	[Ch3]
X_{23}	12	17	2	-1	11	1	$X_{23} \stackrel{(4,5)}{\sim} X_9$	$ K $ defines a $K3$ fibration over \mathbb{P}^1 , the fibres are $\Sigma_{(2,2,2)}$ in \mathbb{P}^5	minimal	[Ed]
X_{24}	12	18	3	-2	13	2	$X_{24} \stackrel{(3,6)}{\sim} X_4$	$ K $ defines an elliptic fibration (in plane cubics) over \mathbb{P}^2	minimal elliptic	[Ed]
X_{25}	13	18	0	1	9	$-\infty$	$X_{25} \stackrel{(5,5)}{\sim} X_{20}$	log-gen type		[BOSS2]
X_{26}	13	18	1	0	10	0	$X_{26} \stackrel{(5,6)}{\sim} X_{29}$	$ K + H $ defines a birational map onto a Calabi-Yau 3-fold $Y \subset \mathbb{P}^9$ with $c_3(Y) = -64$	blown up Calabi-Yau	
X_{27}	13	19	0	1	11	$-\infty$	$X_{27} \stackrel{(4,5)}{\sim} X_6$	$ K + H $ defines a birational map onto $\mathbb{G}(1, 5) \cap \mathbb{P}^9$; birat. to cubic 3-fold in \mathbb{P}^4 [Fa2]	unirational not rational	[Is] [Ch3]
X_{28}	14	22	1	0	14	0	$X_{28} \stackrel{(5,5)}{\sim} X_{17}$	$ K + H $ defines a birational map onto $\mathbb{G}(1, 6) \cap \mathbb{P}^{13}$;	birational Calabi-Yau	[Ch3]
X_{29}	17	32	1	0	24	0	$X_{29} \stackrel{(5,5)}{\sim} X_1 \cup \bigcup_{i=1}^5 \mathbb{P}^3$	$K = K_1 + K_2$, $H^2K_1 = 10$; $ H + K + K_1 $ birat. onto a c.i. $(1, 1, 1, 1, 1)^2$ in $\bigtimes_{i=1}^5 \mathbb{P}^1 \subset \mathbb{P}^{31}$	birational Calabi-Yau (elliptic)	
X_{30}	18	35	0	2	26	$-\infty$		log-general type	not rational	

- Remark 7.4.** (i) The classification of smooth 3-folds in \mathbb{P}^5 is complete up to degree 11 and almost complete in degree 12.
 (ii) Some of the information in Table 7.3 is new. It can be obtained along the lines of section 5 (compare Example 5.8).
 (iii) In order to construct X_{21} via a liaison $X_{21} \stackrel{(5,5)}{\sim} X_{15} \cup X_1$, one has to choose X_{15} and X_1 in a special position. \square

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