Homework 8

Section 2.3

See pg302 for 2c, 2f, 6, 7.

2(d). Let R be a relation on $X \times Y$. The complement, R^c, of R is also a relation on $X \times Y$. Suppose that $(a, b) \in R^c$. If (b, a) is not in R^c, then $(b, a) \in R$. Since R is symmetric, we have $(a, b) \in R$. Thus (a, b) is not in R^c which is a contradiction to our assumption. Therefore $(b, a) \in R^c$ and R^c is symmetric.

2(e). Let R be a relation on $X \times Y$. The reverse $R^r \subset Y \times X$ of R is defined to be the set

$$\{(a, b)|(b, a) \in R\}.$$

If $(a, b) \in R^r$ and $(b, c) \in R^r$ then, by definition, $(b, a) \in R$ and $(c, b) \in R$. By assumption, R is transitive and $(c, b), (b, a)$ are both in R, so we have $(c, a) \in R$. Thus $(a, c) \in R^r$, which concludes the proof.

4. Take a point in each country and join two points by an edge if the two countries represented by those points have a common border. The result is a planar graph. See the following website for further information about the Four Color Theorem:

http://www.math.gatech.edu/˜thomas/FC/fourcolor.html

Section 4.1 1,2,3 see pp. 307-308

– Let X= 1,2,3,4,5. For each part, define a relation R on X satisfying

1. R is reflexive and symmetric, but not transitive

One example that works here is xRy iff $|x - y| \leq 1$. Reflexivity and symmetry are easy to check, and so is the failure of transitivity: 1R2 and 2R3 are true, but 1R3 is false.

2. R is symmetric and transitive, but not reflexive

Suppose xRy for any x,y in X. Then, by symmetry, yRx, so by transitivity xRx. If this were true of every x, then R would be reflexive, so there must be at least one z in X so that zRy is never true for any y in X. One example is the empty relation: xRy is never true. Another is the relation given by xRy iff neither x nor y equals 5.

3. R is reflexive, symmetric, transitive, and weakly anti-symmetric

Suppose xRy. Then symmetry gives us yRx, and weak antisymmetry gives us x=y, so xRy implies x=y. Reflexivity tells us that x=y implies xRy. Thus, xRy iff x=y, so R must be =, and we see that this works: = is reflexive, symmetric, transitive, and weakly antisymmetric.

– Let M be the relation on the real numbers $\mathbb{R}$ defined as follows: for all $x, y \in \mathbb{R}$, xMy if and only if $x - y$ is an integer.

M is an equivalence relation: $x - x = 0 \in \mathbb{Z}$ so M is reflexive, if $x - y \in \mathbb{Z}$ then $y - x \in \mathbb{Z}$ so M is symmetric, and if $x - y \in \mathbb{Z}$ and $y - z \in \mathbb{Z}$, then $x - z = (x - y) + (y - z) \in \mathbb{Z}$ so M is transitive.

The equivalence classes of M correspond to all possible fractionary parts of real numbers. Thus there is an equivalence class for every number $t \in [0, 1)$. Equivalently the collection of equivalence classes of M form a circle (the real line wraps infinitely many times around it).