Homework 2

Section 1.3

1, 3, 4: See pg 297

6. If n is not a prime, then there are two positive integers a, b, with $a > 1$, $b > 1$, such that $n = ab$. Thus $2^n - 1 = 2^{ab} - 1 = (2^a)^b - 1 = (2^a - 1)((2^a)^{b-1} + \cdots + 1)$. Since a and b are both greater than 1, we heave that $2^a - 1$ and $(2^a)^{b-1} + \cdots + 1$ are also greater than 1. Thus $2^n - 1$ is not a prime, a contradiction. Therefore, n must be prime.

7. Let $n = 2^a b$ where b is an odd number and a is a nonnegative integer. Assume $b > 1$. Then $2^n + 1 = 2^{2^a b} + 1 = (2^{2^a})^b + 1 = (2^{2^a} + 1)((2^{2^a})^{b-1} - (2^{2^a})^{b-2} + \cdots + 1)$. Since $2^{2^a} + 1 > 1$ and $b > 1$, we get $(2^{2^a})^{b-1} - (2^{2^a})^{b-2} + \cdots + 1 > 1$. Thus $2^n + 1$ is not a prime, and so so $n = 2^a$.

8. Assume there are only finitely many primes of the form $4k + 3$. Denote them $p_1 = 3, p_2 = 7, \ldots, p_n$, where p_n is the largest prime of form of $4k + 3$. Let now $N := 4(p_2 \times \cdots \times p_n) + 3$. It is not difficult to see that N is not divisible by any of the primes $p_1, \ldots, p_n$. On the other hand any odd integer is either of the form $4k + 1$ or $4k + 3$, and since N is not divisible by any prime $p_1, p_2, \ldots, p_n$, we may write $N = (4a_1 + 1)(4a_2 + 1)\cdots(4a_m + 1)$ for suitable positive integers $a_1, ..., a_m$. Expanding the right hand side we may write $(4a_1 + 1)(4a_2 + 1)\cdots(4a_m + 1) = 4b + 1$ for some integer b. Thus

$$4(p_2 \times \cdots p_n) + 3 = 4b + 1,$$

an hence

$$2 = 4(b - p_2 \times \cdots \times p_n).$$

This implies that

$$1 = 2(b - p_2 \times \cdots \times p_n),$$

which is a contradiction since the right hand side is an even number and the left hand side is an odd number. Therefore there exist infinitely many prime numbers of the form $4k + 3$.

Section 1.4

1, 2. See pgs 297-298.