Homework 10

Section 4.3

2. By definition, $(ab)^{-1}$ is the UNIQUE element such that $(ab)(ab)^{-1} = 1$. But $(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1} = aa^{-1} = 1$. So, by uniqueness of inverses, $(ab)^{-1} = b^{-1}a^{-1}$.

For the example for the second part, see p. 309.

3. To show that G is abelian, we must show that $ab = ba$ for all a and b in G. Now, if a and b are in G, so is their product ab, so our hypothesis tells us that ab is its own inverse; that is $ab = (ab)^{-1}$. But $ab(ba) = a(bb)a = aa = 1$, so by uniqueness of inverses $(ab)^{-1} = ba$. Thus, $ab = ba$ for all a and b in G, so G must be abelian.

5. For the existance of our unit A, see p. 310. Now, every element of G is of the form kA, where k is a non-zero real number and A is our unit. The product of two such elements is given by $(xA)(yA) = (xy)A$. Now, notice that the A is completely extraneous in this equation; dropping it, we see that G has exactly the structure of the non-zero real numbers with multiplication, which we know is a group.

6. Symmetries of the square do not take non-adjacent pairs of vertices to adjacent pairs. This eliminates 2/3 of the possible permutations.

8. see p. 310.

Section 4.4 1,3 - see p. 310

Problem: Describe the group of symmetries of a regular hexagon.

Solution: Label the vertices of the hexagon a,b,c,d,e,f. Now, imagine that you are picking a regular hexagon up off of a table and placing it down onto the hexagonal space where it came from. If we know where to place vertex a, then there are just two possibilities, corresponding to whether we put the hexagon down “right-side-up” or “up-side-down”. If we know where to put vertex a AND whether the hexagon is right-side up or upside down, we have completely specified the placement of the hexagon. Any symmetry of the hexagon is essentially such an operation, so we can look at all of these ways of picking the hexagon up and putting it back down.

First, consider the symmetries that leave the hexagon right-side-up. Clearly, all we are doing is picking the hexagon up, rotating it some angle, and setting it back down. There are six such rotations, corresponding to the six possible positions of vertex a. Let R be the rotation that places a where b used to be. Trivial examination shows that this rotation, iterated over and over, will produce all the rotations; R^2 takes a to c, R^3 takes a to d, etc. So there are six rotations, all generated by R.

Now, consider the operation that flips the hexagon over, taking a to a; let us call this F, for flip. Now, any final position that leaves the hexagon upside down can be seen as a flip followed by an appropriate rotation. Thus, every symmetry can be expressed in the form $F^m R^n$ for some m and n. Further, $F^2 = R^6 = id$, so this tells us that there are a total of 12 distinct symmetries as m ranges over 0,1 and n ranges over 0,1,2,3,4,5.

To completely finish characterizing our group, we need to know how to multiply two elements. To do this, we need to know how F interacts with R. Examination reveals that $FR = R^{-1}F$. With this rule, we know all we need to know about this group; the product of $F^k R^l$ and $F^m R^n$ will be $F^{m+k} R^{n+(-1)^m l}$.