Solutions for Homework 1 (sketch)

HW 1.1

1. Since you can find the answers from the textbook, I will only solve (vi) which looks slightly more complicated.

$$\begin{pmatrix} 1 & 0 & | & 6499 \\ 0 & 1 & | & 4288 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & | & 2211 \\ 0 & 1 & | & 4288 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & | & 2211 \\ -1 & 2 & | & 2077 \end{pmatrix} \to$$

$$\begin{pmatrix} 2 & -3 & | & 134 \\ -1 & 2 & | & 2077 \end{pmatrix} \to \begin{pmatrix} 2 & -3 & | & 134 \\ -31 & 47 & | & 67 \end{pmatrix} \to \begin{pmatrix} 64 & -97 & | & 0 \\ -31 & 47 & | & 67 \end{pmatrix}$$

Thus we have $(-31)(6499) + (47)(4288) = 67$.

2. We find the g.c.d of 6 and 14 first. It is not difficult to see that

$$(1)(14) + (-2)(6) = 2 \cdots (a).$$

The g.c.d of 6, 14, 21 is same as the g.c.d of 2, 21. Again, we get

$$(1)(21) + (-10)(2) = 1 \cdots (b).$$

Substituting (a) into (b), we have

$$(1)(21) + (-10)(14 + (-2)(6)) = 1.$$

. Simplifying it we get

$$(20)(6) + (-10)(14) + (1)(21) = 1$$

and thus 1 is the g.c.d of 6, 14, 21.

5. Take $a = b = c \neq 1$.

6. Since $(a,c) = 1 = (b,c)$ there exist integers p,q,r,s such that $pa + qc = 1$, $rb + sc = 1$. Multiplying the two equations, we get $(pr)(ab) + pasc + rbqc + qsc^2 = 1 \Rightarrow (pr)(ab) + (pas + rbq + qsc)c = 1$. Thus $(ab,c) = 1$.

7. See pg 296.

8. See pg 296.

HW 1.2

1. See pg 296

2. When $n = 1$, L.H.S=$1^2 = 1$, while R.H.S=$\frac{1(1+1)(2+1)}{6} = 1$=L.H.S. Assume when $n = k$, that $1 + 2^2 + \cdots k^2 = \frac{k(k+1)(2k+1)}{6}$. Then when $n = k+1$, L.H.S=$1 + 2^2 + \cdots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = (k+1)[\frac{k(2k+1)}{6} + \frac{6(k+1)}{6}] = \frac{k+1}{6}[2k^2 + 7k + 6] = \frac{(k+1)(k+2)(2(k+1)+1)}{6}$ =R.H.S.

3. Let F_k be the kth-term of the Fibonacci sequence. Then $(F_1, F_2) = (1,1) = 1$. Assume $(F_k, F_{k+1}) = 1$. The main observation is that either by the lemma used to prove the correctness of the Euclidean Algorithm, or just directly we have $(F_{k+1}, F_{k+1} + F_k) = (F_{k+1}, F_k)$. To complete the induction, it is easy now to see that $(F_{k+1}, F_{k+2}) = (F_{k+1}, F_{k+1} + F_k) = (F_{k+1}, F_k) = 1$.

5. When $n = 1$, L.H.S=$\frac{1}{3}$, R.H.S=$\frac{1}{2+1} = \frac{1}{3}$ =L.H.S. Assume when $n = k$, that $\frac{1}{3} + \cdots \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}$. Then when $n = k+1$, L.H.S=$\frac{1}{3} + \cdots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)} = \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} = \frac{k(2k+3)+1}{(2k+1)(2k+3)} = \frac{2k^2+3k+1}{(2k+1)(2k+3)} = \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} = \frac{k+1}{2k+3}$=R.H.S.

6. See pg 297.

7. Base case - take $n = 1$. Then $\frac{1-x^2}{1-x} = 1 + x^1$, so our base case is satisfied.

Inductive step - assume true for n. Then

$$1 + x + \ldots + x^n + x^{n+1} = \frac{1 - x^{n+1}}{1 - x} + x^{n+1}$$

so, multiplying top and bottom and adding, we get

$$1 + \ldots + x^{n+1} = \frac{1 - x^{n+1} + x^{n+1} - x^{n+2}}{1 - x} = \frac{1 - x^{n+2}}{1 - x}$$

which concludes the proof.

8. (i) When $n = 1$, $1^5 - 1 = 0$ and 0 is divisible by 5. Assume when $n = k$, that $5|k^5 - k$. Then when n=k+1, $(k+1)^5 - (k+1) = k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1 = (k^5 - k) + 5(k^4 + 2k^3 + 2k^2 + k)$ is divisible by 5.

(ii) When $n = 1$, $3^2 - 1 = 8$ is divisible by 8. Assume when $n = k$, that $8|3^{2k} - 1$. When $n = k+1$, $3^{2(k+1)} - 1 = 3^2 \cdot 3^{2k} - 1 = 9 \cdot 3^{2k} - 1 = 9 \cdot 3^{2k} - 9 + 9 - 1 = 9(3^{2k} - 1) + 8$ is divisible by 8.

9. Base Cases (note: as the induction requires two prior cases, we must check at least two consecutive base cases; this is actually the most important trick in this problem):

$$2^0 x_0 = 2 = 1 + 1 = (5 + \sqrt{13})^0 + (5 - \sqrt{13})^0$$

$$2^1 x_1 = 10 = 5 + 5 = 5 + \sqrt{13} + 5 - \sqrt{13}$$

Inductive Step; assume for n, n+1, prove for n+2 (see the two base cases?):

$$\begin{aligned}
2^{n+2}x_{n+2} &= 2^{n+2}(5x_{n+1} - 3x_n) \\
&= (10)(2^{n+1}x_{n+1}) - (12)(2^n x_n) \\
&= 10(5+\sqrt{13})^{n+1} + 10(5-\sqrt{13})^{n+1} - 12(5+\sqrt{13})^n - 12(5-\sqrt{13})^n \\
&= (50+10\sqrt{13}-12)(5+\sqrt{13})^n + (50-10\sqrt{13}-12)(5-\sqrt{13})^n \\
&= (38+10\sqrt{13})(5+\sqrt{13})^n + (38-10\sqrt{13})(5-\sqrt{13})^n \\
&= (5+\sqrt{13})^2(5+\sqrt{13})^n + (5-\sqrt{13})^2(5-\sqrt{13})^n \\
&= (5+\sqrt{13})^{n+2} + (5-\sqrt{13})^{n+2}
\end{aligned}$$

This completes the proof.