SKETCH OF SOLUTIONS (HOMEWORK VIII)

1.- a) $\{1,5\}$ b) $\{1,2,4,5,7,8\}$ c) $\{1,3,7,9\}$ d) $\{1,3,5,9,11,13\}$ e) $\{1,3,5,7,9,11,13,15,17\}$ f) $\{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16\}$

6.- Notice that $\Phi(10) = 4$ and $999,999 = 4(249,999)+3$. Using Euler's theorem we get that the last decimal digit is 3.

8.- Using Fermat's theorem we get $a^7 \equiv a \mod 7$. We only need to show that $a^9 \equiv a \mod 9$ (since $9 \cdot 7 = 63$). If $9 \mid a$ we have $0 \equiv 0 \mod 9$. If $3 \nmid a$ then $(a,9) \equiv 1$. But since $\Phi(9) = 6$ we have $a^6 \equiv 1 \mod 9$ i.e. $a^7 \equiv a \mod 9$.

12.- Notice that x is a solution by Euler's theorem, and it is unique by the Chinese remainder theorem.

Section 7.1

1.- a) Yes: $f(mn) = 0 = 0 \cdot 0 = f(m)f(n)$, b) No: $f(2\cdot 2) = 2 \neq 4 = f(2)f(2)$, c) No: $f(2\cdot 3) = 3 \neq \frac{2}{2}\frac{3}{2} = f(2)f(3)$, d) No: $\log(4) > 1$ (since $4 > e$) but $\log(2)\log(2) < 1$, e) Yes: $f(mn) = (mn)^2 = m^2n^2 = f(m)f(n)$, f) No: $f(2\cdot 2) = 4! \neq 4 = f(2)f(2)$, g) No: $f(1\cdot 1) = 2 \neq 2\cdot 2 = f(1)f(1)$, h) No: $f(4) = 4^4 = 256 \neq 16 = f(2)f(2)$, i) Yes: $f(mn) = \sqrt{mn} = \sqrt{m}\sqrt{n} = f(m)f(n)$

2.- c) $\Phi(1001) = \Phi(7\cdot 11\cdot 13) = 6\cdot 10\cdot 12 = 720$ e) Using theorem 7.5 we get $\Phi(10!) = 10!(1-1/2)(1-1/3)(1-1/5)(1-1/7)$ (All the prime factors must be all the primes less than or equal to 10). $= 829,440$

3.- They all equal 2592

5.- We know $6 = \Phi(n) = \Pi_{j=1}^{k} p_j^{a_j-1}(p_j-1)$. But since $\Phi(p) \geq 4$ for all primes greater than or equal to 5, we must have $k \leq 2$.(Otherwise three or more prime factors would give a value of $\Phi(n)$ greater than or equal to $4\cdot 2$) If $k = 1$ then $p_1^{a_1-1}(p_1-1) = 6$ Notice that we only need to try values of p_1 between 2 and 7. The solutions for this case are $n = 7, 9$. If $k = 2$ then the solutions are $n = 14, 18$.

14.- Suppose $k\Phi(n) = kn(p_1-1)/p_1 \cdots (p_r-1)/p_r = n$. Then $k = p_1/(p_1-1)\cdots p_r/(p_r-1)$ is an integer. There can be at most one even number among the p_i (because 2 is the only even prime), so there can be at most one odd prime among the p_i (since k is an integer). The only possible values for n are $n = 1, 2^{a_1}, 2^{a_1}3^{a_2}$ with $a_1, a_2 \geq 1$

35.- a) If either $m > 1$ or $n > 1$ then $mn > 1$ and one of $i(m)$ or $i(n)$ is equal to zero. Then $i(mn) = 0 = i(m)i(n)$. Otherwise, $m = n = 1$ and we have $i(mn) = 1 = 1\cdot 1 = i(m)i(n)$.
b) $(i * f)(n) = \sum_{d|n} i(d)f(n/d) = i(1)f(n) = f(n)$ by the definition of i. $f * i(n) = \sum_{d|n} f(d)i(n/d) = f(n)$ also by the definition of i