SKETCH OF SOLUTIONS (HOMEWORK I)

2.- Define the set $S \subset \mathbb{N}$ by $S = \{n \in \mathbb{N} \mid n = a - bk,\ k \in \mathbb{Z}\}$. S is not empty since the hypotheses on a imply $a \in S$ (taking $k = 0$) therefore S must have a minimum element by the well ordering principle.

4.- a) True: (Proof by contradiction) Let t be irrational and $\frac{a}{b}$ be a rational number ($a, b \in \mathbb{Z}$, $b \neq 0$). Suppose $t + \frac{a}{b}$ is rational, that is, $\frac{t}{1} + \frac{a}{b} = \frac{bt+a}{b} = \frac{p}{q}$ with $p, q \in \mathbb{Z}$, $q \neq 0$. Then we get that

$$t = \frac{bp - qa}{qb}$$

but this means that t is rational! (a contradiction). Therefore $t + \frac{a}{b}$ is irrational.

b) False: $\sqrt{2} - \sqrt{2} = 0 \in \mathbb{Q}$

c) False: $0 \cdot \sqrt{2} = 0 \in \mathbb{Q}$

d) False: $\sqrt{2} \cdot \sqrt{2} = 2 \in \mathbb{Q}$

26.- $\sum_{k=2}^{n} \frac{1}{k^2} = \frac{1}{2} \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k+1} \right)$ notice that in this last sum all terms cancel each other out, except the first two terms. Therefore we get:

$$\sum_{k=2}^{n} \frac{1}{k^2} = \frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}\right] = \frac{1}{2}\left[\frac{3}{2} - \frac{2n+1}{n(n+1)}\right]$$

27.- Notice $(k+1)^3 - k^3 = 3k^2 + 3k + 1$ thus $k^2 = \frac{1}{3}[(k+1)^3 - k^3 - 3k - 1]$. Adding over k we get:

$$\sum_{k=1}^{n} k^2 = \frac{1}{3}\left[\sum_{k=1}^{n} \left((k+1)^3 - k^3\right) - 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1\right] = \frac{1}{3}\left[(n+1)^3 - 1 - 3\frac{n(n+1)}{2} + n\right]$$

(Simplifying the expression we get: $\frac{n(n+1)(2n+1)}{6}$)

Section 1.2

5.-

$$A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

Proof by induction:

Base: When $n = 1$ this is just the definition of A

Inductive step: Suppose $A^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ We must show then that $A^{k+1} = \begin{pmatrix} 1 & k+1 \\ 0 & 1 \end{pmatrix}$. But $A^{k+1} = A \cdot A^k = A \cdot \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ (By the induction hypothesis) Multiplying we get:

$$A^{k+1} = A \cdot A^k = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & k+1 \\ 0 & 1 \end{pmatrix}$$

10.- **Base:** $\sum_{j=1}^{1}(-1)^{j-1}j^2 = 1 = (-1)^0\frac{1(1+1)}{2}$
Inductive step: Suppose $\sum_{j=1}^{k}(-1)^{j-1}j^2 = (-1)^{k-1}\frac{k(k+1)}{2}$ Then $\sum_{j=1}^{k+1}(-1)^{j-1}j^2 = \sum_{j=1}^{k}(-1)^{j-1}j^2 + (-1)^k(k+1)^2 = (-1)^{k-1}\frac{k(k+1)}{2} + (-1)^k(k+1)^2 = (-1)^k\frac{(k+1)(k+2)}{2}$

16.- **Base:** $H_{2^1} = \sum_{j=1}^{2}\frac{1}{j} = 1 + \frac{1}{2} \leq 1 + 1$
Inductive step: Suppose $H_{2^k} = \sum_{j=1}^{2^k}\frac{1}{j} \leq 1 + k$. Then $H_{2^{k+1}} = \sum_{j=1}^{2^{k+1}}\frac{1}{j} = \sum_{j=1}^{2^k}\frac{1}{j} + \sum_{j=2^k+1}^{2^{k+1}}\frac{1}{j} \leq 1 + k + \sum_{j=2^k+1}^{2^{k+1}}\frac{1}{j}$ so we only need to make sure the last sum is less than 1. But for this last sum we have $\frac{1}{j} \leq \frac{1}{2^k}$ (by the range of the indices) therefore

$$\sum_{j=2^k+1}^{2^{k+1}}\frac{1}{j} \leq \sum_{j=2^k+1}^{2^{k+1}}\frac{1}{2^k} = \frac{1}{2^k}\sum_{j=2^k+1}^{2^{k+1}}1 = \frac{1}{2^k}(2^{k+1} - (2^k+1) + 1) = 1$$

30.- **Base:** $2^5 = 32 > 25 = 5^2$
Inductive step: Suppose $2^k > k^2$ Then $2^{k+1} = 2 * 2^k > 2k^2 = k^2 + k^2 > k^2 + 2k + 1 = (k+1)^2$ The last inequality is a consequence of the following inequality: $k^2 > 2k+1$ for $k > 3$ The proof of this inequality goes as follows $k > 3 \Rightarrow k^2 > 3k = 2k + k > 2k + 1$

Section 1.3

4.- $f_2 + f_4 + \ldots + f_{2n} = f_{2n+1} - 1$ The proof is by induction over n
Base: $f_2 = f_3 - f_1 = f_3 - 1$
Inductive step: Suppose $f_2 + f_4 + \ldots + f_{2k} = f_{2k+1} - 1$ then $f_2 + f_4 + \ldots + f_{2k} + f_{2(k+1)} = f_{2k+1} - 1 + f_{2k+2} = f_{2k+3} - 1 = f_{2(k+1)+1} - 1$

8.- By induction on k
Base: $f_1 f_2 = 1 * 1 = f_2^2$
Inductive step: Suppose $f_1f_2 + \ldots f_{2k-1}f_{2k} = f_{2k}^2$ then $f_1f_2 + \ldots f_{2k-1}f_{2k} + f_{2k}f_{2k+1} + f_{2(k+1)-1}f_{2(k+1)} = f_{2k}^2 + f_{2k}f_{2k+1} + f_{2k+1}f_{2k+2} = f_{2k}(f_{2k} + f_{2k+1}) + f_{2k+1}f_{2k+2} = f_{2k}f_{2k+2} + f_{2k+2}f_{2k+1} = f_{2k+2}(f_{2k} + f_{2k+1}) = f_{2k+2}f_{2k+2} = f_{2(k+1)}^2$

22.- **Note: There is a mistake on the book. The last term of the sum should be**

$$\binom{\lceil \frac{n}{2} \rceil}{\lfloor \frac{n}{2} \rfloor}$$

Base: $\binom{1}{0} = 1 = f_2$
Inductive step: Suppose $\binom{n}{0} + \binom{n-1}{1} + \ldots + \binom{\lceil \frac{n}{2} \rceil}{\lfloor \frac{n}{2} \rfloor} = f_{n+1}$ for all $n \leq k$ then let $x := \binom{n}{0} + \binom{n-1}{1} + \ldots + \binom{\lceil \frac{n+1}{2} \rceil}{\lfloor \frac{n+1}{2} \rfloor}$ (that is, x is the sum corresponding to $k = n + 1$) Then, using **Pascal's Identity** (Theorem B.2. of the book) we get that

$$x - f_{n+1} = \left[\binom{n+1}{0} - \binom{n}{0}\right] + \left[\binom{n}{1} - \binom{n-1}{1}\right] + \ldots + = 0 + \binom{n-1}{0} + \binom{n-2}{1} + \ldots +$$

which by the inductive hypothesis equals f_n therefore we get

$$x - f_{n+1} = f_n$$

But by the definition of the Fibonacci numbers this means $x = f_{n+2}$