MAT 311: Number Theory
Spring 2006

Solutions to HW6

1. (Davenport, pp.219, ex. 2.21) We will prove that $d \mid n$ implies $\varphi(d) \mid \varphi(n)$. Let the prime factorization of d be $d = \prod_{i=1}^{K} p_i^{\alpha_i}$. Then the prime factorization of n is of the form
$$n = \prod_{i=1}^{K} p_i^{\alpha_i+\beta_i} \cdot \prod_{i=1}^{N} q_i^{\gamma_i}.$$
Then
$$\begin{aligned}\varphi(n) &= \prod_{i=1}^{K} p_i^{\alpha_i+\beta_i}\left(1-\frac{1}{p_i}\right) \cdot \prod_{i=1}^{N} q_i^{\gamma_i}\left(1-\frac{1}{q_i}\right) \\ &= \left(\prod_{i=1}^{K} p_i^{\alpha_i}\left(1-\frac{1}{p_i}\right)\right) \cdot \left(\prod_{i=1}^{K} p_i^{\beta_i}\right) \cdot \left(\prod_{i=1}^{N} q_i^{\gamma_i}\left(1-\frac{1}{q_i}\right)\right) \\ &= \varphi(d) \cdot \prod_{i=1}^{K} p_i^{\beta_i} \cdot \varphi\left(\prod_{i=1}^{N} q_i^{\gamma_i}\right) = \varphi(d) \cdot \text{some integer.}\end{aligned}$$
Now, that all the factors are integers, we deduce that $\varphi(d)$ divides $\varphi(n)$.

2. (Davenport, p.219, ex. 2.22) We will show that for any prime p different from 2 and 5, there are infinitely many numbers of the form $\sum_{i=0}^{n} 10^i$ (called *repeated units*, or *repunits*, for short) divisible by p. If $p \neq 2, 5$, then $(10, p) = 1$, and so, by FlT, we have $10^{p-1} \equiv 1 \mod p$. This implies also that $10^{(p-1)\cdot k} \equiv 1 \mod p$ for any $k \geq 0$. So, p divides $10^{(p-1)\cdot k} - 1 = 9 \cdot (1 + 10 + 10^2 + \cdots + 10^{(k\cdot(p-1))-1})$. If p is not 3, then p should divide the second factor, which is a repunit for any k, done. If $p = 3$, then the repunit $1 + 10 + \cdots + 10^{3n-1}$ is divisible by 3, since $10 \equiv 1 \mod 3$. This completes the proof.

3. (a) We will show that $\tau(n)$ is odd iff n is a perfect square. Indeed, if the prime factorization of n is $n = \prod_{i=1}^{K} p_i^{\alpha_i}$, then $\tau(n) = \prod_{i=1}^{K}(\alpha_i + 1)$. So, $\tau(n)$ is odd iff all $\alpha_i + 1$ are odd iff all α_i are even iff n is a perfect square.
   (b) We will show that $\sigma(n)$ is odd iff n is a perfect or twice a perfect square. Let $n = \prod_{i=1}^{K} p_i^{\alpha_i}$, as above. Then $\sigma(n) = \prod_{i=1}^{K}\left(\sum_{j=0}^{\alpha_i} p_i^j\right)$. So, $\sigma(n)$ is odd iff each $\sum_{j=0}^{\alpha_i} p_i^j$ is odd. But for if p_i is odd then this sum is $\equiv \alpha_i + 1 \mod 2$; and if $p_i = 2$ then this sum is definitely odd. So, this sum is odd iff for any i we have either (a) p_i is odd and α_i is even, OR, (b) $p_i = 2$ (doesn't matter what the power of 2 is). Therefore, $\sigma(n)$ is odd iff n is a perfect square or twice a perfect square.

4. (a) A short computer program can find the six smallest abundant number within fractions of a second: 12, 18, 20, 24, 30, 36.
   (b) We will show that a multiple of an abundant or a perfect number (other than the perfect number itself) is abundant. Let $\sigma(n) \geq 2n$. Then we have
$$\sigma(nk) = \sum_{x \mid nk} x \geq \sum_{y \mid n} \sum_{z \mid k} yz$$
   because $y \mid n$ and $z \mid k$ imply that $yz \mid nk$ (so that the latter sum is taken over fewer divisors). Then
$$\sigma(nk) \geq \sum_{y \mid n} \sum_{z \mid k} yz = \sum_{y \mid n} y \sum_{z \mid k} z = \sigma(n) \cdot \sigma(k) > (2n)k$$
   whenever $k > 1$ (since $\sigma(k) > 1$ if $k > 1$).

(c) We will prove that $2^{m-1}(2^m-1)$ is abundant if 2^m-1 is composite. Indeed, if there is an integer $n \neq 1, 2^m-1$ dividing 2^m-1, then $\sigma(2^m-1) > (2^m-1)+1 = 2^m$. On the other hand, $\sigma(2^{m-1}(2^m-1)) = \sigma(2^{m-1})\sigma((2^m-1))$ (since these two factors are obviously coprime) $= (2^m-1)\sigma(2^m-1) > (2^m-1)2^m = 2(2^{m-1}(2^m-1))$, as required.

5. (a) $\Lambda(n)$ is not multiplicative simply because $\Lambda(1) = 0 \neq 1$.

   (b) We will prove the formula $\sum_{d\,|\,n} \Lambda(d) = \log n$. Indeed,

$$\sum_{d\,|\,n} \Lambda(d) = \sum_{p^i\,|\,n} \Lambda(p^i) = \sum_{p^i\,|\,n} \log p = \log \prod_{p^i\,|\,n} p = \log n.$$

6. (a) We will show that the Liouville function $\lambda(n)$ is multiplicative. Given two coprime numbers n, m. Say $n = \prod_{i=1}^N p_i^{\alpha_i}$ and $m = \prod_{j=1}^M q_j^{\beta_j}$. Note that $p_i \neq q_j$ for any i, j since $(n, m) = 1$. Now, $\lambda(n) = (-1)^N$ and $\lambda(m) = (-1)^M$, and moreover $\lambda(nm) = (-1)^{N+M}$ since nm is a product of all $p_i^{\alpha_i}$'s and $q_j^{\beta_j}$'s so that nm has $N+M$ (distinct) prime factors. Hence λ is multiplicative.

   (b) We will show that the convolution inverse of $\lambda(n)$ is the characteristic function of the squarefree numbers (which is the function μ^2, where μ is the Möbius function). So we have to prove that $(\lambda * \mu^2)(1) = 1$ and $(\lambda * \mu^2)(n) = 0$ for $n \geq 2$. Since both λ and μ are multiplicative, so is $\lambda * \mu^2$. Let $n \geq 2$, and write $n = \prod_{i=1}^N p_i^{\alpha_i}$, $\alpha_i \geq 1$. Then

$$(\lambda * \mu^2)(n) = \prod_{i=1}^{K} (\lambda * \mu^2)(p_i^{\alpha_i}) = \prod_{i=1}^{K} \sum_{d\,|\,p_i^{\alpha_i}} \lambda(d)\mu^2(\frac{p_i^{\alpha_i}}{d}).$$

   But $\mu^2(\frac{p_i^{\alpha_i}}{d})$ is zero except $\frac{p_i^{\alpha_i}}{d} = 1$ or p_i, and in these cases it is equal to 1 (recall that the Möbius function μ is zero for non-squarefree numbers, and $(-1)^k$ if the number is product of k -necessarily distinct- primes). So,

$$(\lambda * \mu^2)(p_i^{\alpha_i}) = \lambda(p_i^{\alpha_i}) \cdot \mu^2(1) + \lambda(p_i^{\alpha_i - 1}) \cdot \mu^2(p_i) = \lambda(p_i^{\alpha_i}) \cdot \lambda(p_i^{\alpha_i-1}) = (-1)^{\alpha_i} + (-1)^{\alpha_i - 1} = 0.$$

   Hence, $(\lambda * \mu^2)(n) = 0$ for $n \geq 2$. On the other hand, since $\mu(1) = \lambda(1) = 1$ we have $(\lambda * \mu^2)(1) = 1$. This completes the proof.

7. We will find a closed form expression for each of the following sums:

   (a) Let $x := \sum_{d\,|\,n} \frac{\mu(d)}{d}$. Hence, $nx = \sum_{d\,|\,n} \mu(d)\frac{n}{d}$. If we denote the identity function by ι (that is, $\iota(n) = n$, $\forall n$), then $nx = (\mu * \iota)(n)$. Now, let $n = \prod p_i^{\alpha_i}$. Then, since $\mu * \iota$ is multiplicative,

$$\begin{aligned}(\mu * \iota)(\prod p_i^{\alpha_i}) &= \prod (\mu * \iota)(p_i^{\alpha_i}) = \prod \left(\sum_{d\,|\,p_i^{\alpha_i}} \mu(d)\iota(\frac{p^{\alpha_i}}{d}) \right) \quad \text{but } \mu(p_i^k) = 0 \text{ for all } k \geq 2, \text{ thus} \\ &= \prod \left(\mu(1)\iota(p_i^{\alpha_i}) + \mu(p_i)\iota(p_i^{\alpha_i - 1}) \right) = \prod (p_i^{\alpha_i} - p_i^{\alpha_i-1}) = \varphi(n).\end{aligned}$$

   Thus, $x = \frac{\varphi(n)}{n}$.

   (b) Let $f(n) = \mu(n)\varphi(n)$. Denote by I the constant function $I(n) = 1$, $\forall n$. Then for $n = \prod p_i^{\alpha_i}$ we have

$$\begin{aligned}\sum_{d\,|\,n} f(d) &= (f * I)(\prod p_i^{\alpha_i}) = \prod (f * I)(p_i^{\alpha_i}) \\ &= \prod \left(\sum_{d\,|\,p_i^{\alpha_i}} \mu(d)\varphi(d)I(\frac{p_i^{\alpha_i}}{d}) \right) = \prod (\mu(1)\varphi(1) \cdot 1 + \mu(p_i)\varphi(p_i) \cdot 1) \\ &= \prod (1 - (p_i - 1)) = \prod (2 - p_i).\end{aligned}$$

(c) Let $g(n) = \mu^2(n)/\varphi(n)$. Then for $n = \prod p_i^{\alpha_i}$ we have

$$\begin{aligned}
\sum_{d\,|\,n} g(n) &= (g * I)(\prod p_i^{\alpha_i}) = \prod (g * I)(p_i^{\alpha_i}) \\
&= \prod \left(\sum_{d\,|\,p_i^{\alpha_i}} \frac{\mu^2(d)}{\varphi(d)} I(\frac{p_i^{\alpha_i}}{d}) \right) = \prod (\frac{\mu^2(1)}{\varphi(1)} \cdot 1 + \frac{\mu^2(p_i)}{\varphi(p_i)} \cdot 1) \\
&= \prod (1 + \frac{1}{p_i - 1}) = \prod (\frac{1}{1 - \frac{1}{p_i}}) = \frac{n}{\varphi(n)}, \quad \textit{cf.}\ \text{(a)}.
\end{aligned}$$

(d) We will show that $\sum_{d\,|\,n} \mu(\frac{n}{d}) \log d = \Lambda(n)$. This follows at once from Möbius inversion formula since $\sum_{d\,|\,n} \Lambda(d) = \log(n)$ by Problem 5. We can also derive it directly: By definition $\sum_{d\,|\,n} \mu(\frac{n}{d}) \log d = (\mu * \log)(n)$. Now, notice that $\mu * I = \epsilon$, where ϵ is the *unit function* (1 if $n = 1$ and 0 otherwise). Thus,

$$((\mu * \log) * I)(n) = (\log *(\mu * I))(n) = (\log *\epsilon)(n) = \log(n).$$

Therefore we conclude that $\sum_{d\,|\,n} (\mu * \log)(d) = ((\mu * \log) * I)(n) = \log(n)$. On the other hand, since $\sum_{d\,|\,n} \Lambda(d) = \log(n)$, we have $(\mu * \log)(d) = \Lambda(d)$ for any d (for instance, by induction).