SKETCH OF SOLUTIONS (HOMEWORK III)

6.1 a) $x = -4515 - 4526k, y = 821 + 823k$
b) $x = 1647 + 3292k, y = -9059 - 18107k$

6.2 a) $x = -53 - 121k, y = 46 + 105k$
b) $x = -4515 - 4526k, y = 821 + 823k$
c) $x = 1647 + 3292k, y = -9059 - 18107k$

6.3 a) $x = -9, y = 1, z = 2$
b) $\gcd(a, b, c) = \gcd(\gcd(a, b), c) = 1$ The general method is to find a solution to $ax+by = \gcd(a, b)$ and then find a solution to the equation $dw + cz = 1$.
c) $x = -32, y = 1, z = 12$

7.2 Let $a = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$ be the prime factorization of a ($\alpha_i > 0, p_i < p_{i+1}$) and $b = q_1^{\beta_1} \cdots p_m^{\beta_m}$ be the prime factorization of b. Since $\gcd(a, b) = 1$ we know that $p_i \neq q_j$ for all i, j. Also, since $a \mid c$ we know that $p_i^{\alpha_i} \mid c \forall i$ and the same holds for b, $q_j^{\beta_j} \mid c \forall j$. Therefore $p_i^{\alpha_i}$ and $q_j^{\beta_j}$ appear in the prime factorization of c.

7.3 a) **Base:** $1 = \frac{1(1+1)}{2}$
Inductive Step Suppose $1 + \ldots + n = \frac{n(n+1)}{2}$ then

$$1 + \ldots + n + (n + 1) = \frac{n(n + 1)}{2} + (n + 1) = \frac{(n + 1)(n + 2)}{2}$$

b) **Base:** $1 = \frac{1(1+1)(2+1)}{6}$
Inductive Step Suppose $1 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$ then

$$1+\ldots+n^2+(n+1)^2 = \frac{n(n + 1)(2n + 1)}{6}+(n+1)^2 = \frac{n(n + 1)(2n + 1) + 6(n + 1)^2}{6} = \frac{(n + 1)(n + 2)(2(n + 1) + 1)}{6}$$

c) **Base:** $a^0 = 1 = \frac{1-a}{1-a}$ $(a \neq 1)$
Inductive Step: Suppose that $1 + a + \ldots + a^n = \frac{1-a^{n+1}}{1-a}$ then

$$1 + a + \ldots a^n + a^{n+1} = \frac{1 - a^{n+1}}{1 - a} + a^{n+1} = frac1 - a^{n+1} + a^{n+1} - a^{n+2}1 - a$$

d) **Base:** $\frac{1}{2} = \frac{1}{2}$
Inductive Step: Suppose that $\frac{1}{1\cdot2} + \cdots + \frac{1}{(n-1)n} = \frac{n-1}{n}$ then

$$\frac{1}{1 \cdot 2} + \cdots + \frac{1}{(n - 1)n} + \frac{1}{n(n + 1)} = \frac{n - 1}{n} + \frac{1}{n(n + 1)} = \frac{n^2}{n(n + 1)} = \frac{n}{n + 1}$$

7.4 a) $\mathbb{E}-$primes are numbers of the form $2q$ where q is odd.
b) Suppose every even number up to $2n$ can be factored as a product of $\mathbb{E}-$primes,then if $2n + 2 = 2(n + 1)$ is an $\mathbb{E}-$prime we are done, otherwise $2(n+1) = 2n_1 \cdot 2n_2$ with $n_1, n_2 < n$ but then the hypothesis implies that n_1 and n_2 can be factored as a product of $\mathbb{E}-$primes and we are done.

c) $2 \cdot 18 = 6 \cdot 6 = 36$, 180 is the smallest with 3 factorizations, 360 is the smallest number with 4 factorizations.

d) Suppose $m = 2^s n$ with $s > 1$ and n odd, then m has a unique factorization as a product of $\mathbb{E}-$primes if either n is prime or m is $\mathbb{E}-$prime.Conversely, suppose that $m = 2^s n$ has a unique factorization as a product of $\mathbb{E}-$primes, then if $s = 1$, m is prime.If $s > 1$ then n must be prime, otherwise if $n = pq$ with $p, q > 1$ we get two different factorizations: $m = (2p)(2q)(2^{s-2})$ and $m = (2pq)(2^{s-1})$