Addition in $\mathbb{N}$

The Peano Axioms give us way to define the natural numbers $\mathbb{N}$ in terms of a starting point (0) and a succsessor function $S(n)$ which gives us the next number after one we know. That is, they tell us that if we've counted to n, we can count one more.

The naturals allow us to count, but we quickly see the need for the operation of addition. If we have two piles of things we have already counted (for example, one pile of 53 rocks, and another pile of 22 rocks), we don't need to count everything again to know the total amount.

Recall that we use $S(n)$ to denote the successor of n.

We want to define the "nth successor" of a number m, that is, what we get after we "add 1" n times, starting with m. This is a little trickier than it sounds, since we have very little to work with. Here is a definition that does the job.

Definition 1. *Recall that $S(n)$ is the successor of $n \in \mathbb{N}$, as in the Peano Axioms. Let $A(m,n)$ be a function which satisfies the following properties:*

(i) $A(0,n) = n$

(ii) $A(m, S(n)) = S(A(m,n))$

This is a *recursive* definition: it has a case that allows us to begin things ($A(0,n) = n$), and a way to move up to larger natural numbers.

This isn't quite the definition we used in class, but let's check that it does what we want. In particular, the second part of the definition seems rather opaque. Before going on, see if you can answer the following, based on the definition: what is $A(3,0)$? What is $A(2,3)$?

Let's establish a couple of helpful facts about $A(m,n)$ for specific cases.

Proposition 2. *For any $m \in \mathbb{N}$, $A(m,0) = n$.*

Proof. Property (i) tells us that $A(0,0) = 0$.

Using (ii), we have $A(1,0) = A(S(0),0) = S(A(0,0))$, but since we know $A(0,0) = 0$, we have shown that $A(1,0) = S(0) = 1$.

More generally, we can apply the above idea to show that whenever we know that for some specific value of k we have $A(k,0) = k$, then property (ii) gives us $A(S(k),0)) = S(A(k,0)) = S(k)$.

In other words, taking $K = \{m \in \mathbb{N} \mid A(m,0) = m\}$, we have shown that $0 \in K$ and also that whenever $k \in K$, it is also true that $S(k) \in K$. Applying the 5th Peano axiom, this means K is all of $\mathbb{N}$. That is, for every natural number n, we have $A(m,0) = m$ as desired. □

Proposition 3. *For any $m \in \mathbb{N}$, $S(m) = A(m,1) = A(1,m)$.*

Proof. Observe that the definition and the previous proposition give us $A(m,1) = A(m,S(0)) = S(A(m,0)) = S(m)$. To see that $A(1,m) = S(m)$, we use induction. From Prop. 2, we have $A(1,0) = 1 = S(0)$, so this holds for 0. Now we must show that if $A(1,k) = S(k)$ for some k, then also $A(1,S(k)) = S(S(k))$. But this is almost immediate:

$$A(1,S(k)) = S(A(1,k)) = S(S(k)).$$

□

We can write $A(m,n)$ more conventionally as $m+n$, but remember, we are trying to understand what "+" means based only what is given in Def. 1, so we can't merely rely on our previous experience with addition. In order to help avoid this, in many cases I will use the notation $A(m,n)$ instead of m,n.

Let's check that "+" has properties we want, when defined in this way.

Proposition 4. *Using the notation $m+n$ to represent $A(m,n)$ above, we have*

(1) Commutativity: for all $m,n \in \mathbb{N}$, $m+n=n+m$.
(2) Associativity: for all $m,n,k \in \mathbb{N}$, $m+(n+k)=(m+n)+k$

Proof. First, let us establish the associativity of $+$, that is, part (2) of the proposition. We let

$$K_{m,n} = \{x \in \mathbb{N} \mid \text{for every } m \text{ and } n \text{ in } \mathbb{N},\ m+(n+x)=(m+n)+x\}.$$

Using the $A(m,n)$ notation, x is in this set if, given any m and n, we have

$$A(m,A(n,x)) = A(A(m,n),x)$$

Using induction, we now show that $K_{m,n} = \mathbb{N}$.

First, we have $0 \in K_{m,n}$, that is, $A(m,A(n,0)) = A(m,n) = A(A(m,n),0)$.
Here we used Proposition 2 twice: to replace $A(n,0)$ by n in the first equality, and then to replace $A(m,n)$ with $A(A(m,n),0)$ in the second.

Now we need to show that if $x \in K_{m_n}$, then $S(x)$ is also in $K_{m,n}$. That is, we want to show that if we have a arbitrary value of k so that $A(m,A(n,k)) = A(A(m,n),k)$, then we also know that $A(m,A(n,S(k))) = A(A(m,n),S(k))$.

For any $m,n \in \mathbb{N}$, we have

$$\begin{aligned}
A(m,A(n,S(k))) &= A(m,S(A(n,k))) && \text{applying Def.1(ii) to } A(n,S(k)) \\
&= S(A(m,A(n,k))) && \text{applying Def.1(ii) again} \\
&= S(A(m,n),k) && \text{since } k \in K_{m,n} \\
&= A((m,n),S(k)) && \text{applying Def.1(ii) yet again}
\end{aligned}$$

as desired.

Applying the induction property of $\mathbb{N}$, we have shown that part (2) holds for all natural numbers.

Now we turn to part (1): Let $P_m = \{n \in \mathbb{N} \mid m+n=n+m\}$. Another way to write this is to say $n \in P_m$ whenever $A(m,n) = A(n,m)$. We want to show P_m is all of $\mathbb{N}$.

Observe that Proposition 2 and Definition 1 tell us that $0 \in P_m$. Also $1 \in P_m$ by Proposition 3.

Now we turn to the inductive step: If we know that $n \in P_m$, we want to show that $S(n)$ is also in P_m. This means we want to show if we know $A(m,n) = A(n,m)$ for some arbitrary n, we also have $A(m,S(n)) = A(S(n),m)$ for that n. So, we have

$$\begin{aligned}
A(m,S(n)) &= S(A(m,n)) && \text{by the definition, part (ii)} \\
&= S(A(n,m)) && \text{since we know } n \in P_m \\
&= A(A(n,m),1) && \text{from proposition 3} \\
&= A(1,A(n,m)) && \text{since } 1 \in P_{A(n,m)} \\
&= A(S(n),m)) && \text{again using the observation}
\end{aligned}$$

which is what we wanted to show. □

We also can show the cancellation law:

Proposition 5. *For any natural numbers m, n, and k, if $x+z=y+z$, then $x=y$.*

I will omit the proof here, but again it can be established using induction on z. The case $z=0$ follows from Definition 1 and Proposition 4, and applying Proposition 3 to $z=1$ becomes the 4th Peano Axiom. The inductive step is similar to the previous arguments.