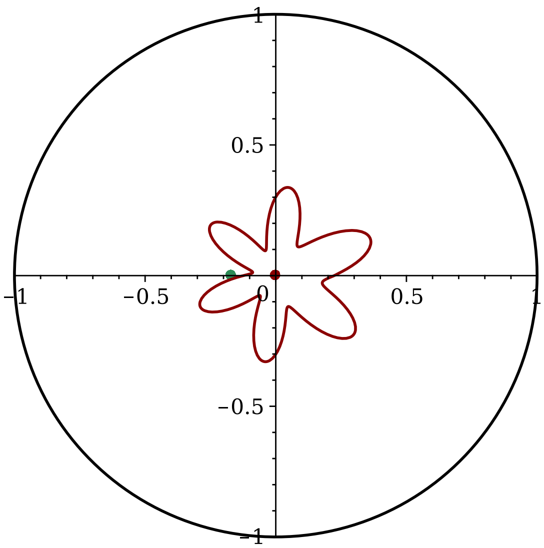
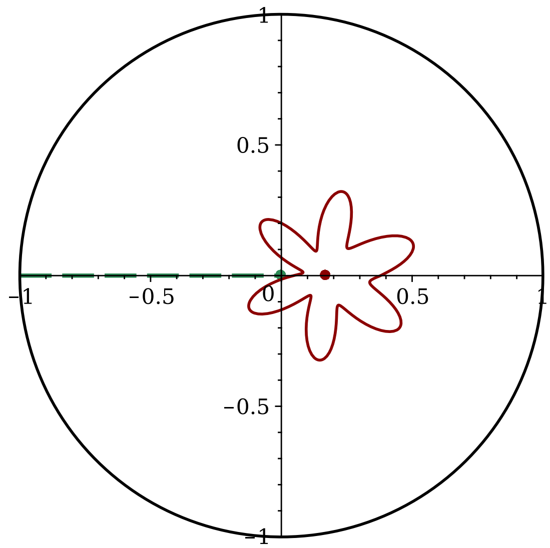
An illustration of the Koebe-Carathéodory square-root trick

As you should recall, we have some simply connected U with a marked point $p \in U$, and some conformal map $f : U \to \mathbb{D}$ with $f(p) = 0$. If $f(U) \neq \mathbb{D}$, the Koebe-Carathéodory square root trick constructs a new map $g : U \to \mathbb{D}$ with the same properties but with $|g'(p)| > |f'(p)|$.

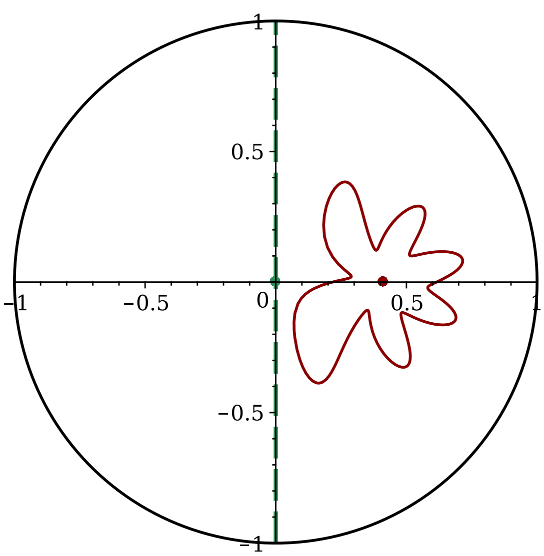
On the left is a simply-connected set $V = f(U)$ (outlined in red) in the unit disk $\mathbb{D}$, with a marked red point $f(p)$ at the origin. A point $a \in \mathbb{D}$ but not in $f(U)$ is chosen (this is shown in green).

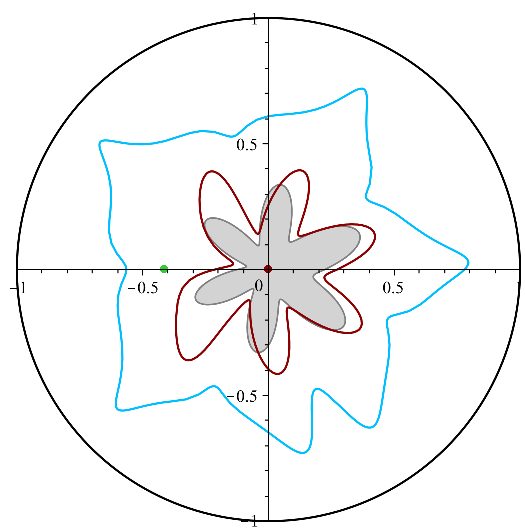
Let $\phi_a(z) = \dfrac{z-a}{1-\bar{a}z}$, which is the automorphism of $\mathbb{D}$ sending a to the origin. On the left is shown $\phi_a(V)$, $\phi_a(0)$ (in red), and the image $\phi_a(a)$ (which is the origin).





Now comes the “trick”: since $0 \notin \phi_a(V)$, there is a holomorphic branch of the square root defined in a neighborhood of V. (Here, it is the principal branch, with the branch cut along the negative real axis.) Let $W = \sqrt{V}$. W is shown on the left below (in red as usual), and then on the right is shown the effect of the composition with the automorphism ϕ_b, where $b = \sqrt{\phi_a(0)}$. This composition gives our new map $g : U \to \mathbb{D}$. In this picture, $g(U)$ outlined in red, and $f(U)$ is shown shaded in gray for comparison, with the point $\phi_b(0)$ in light green. Also shown (in blue) is the result of re-applying the process six more times, choosing points a_i of various arguments.





Scott Sutherland, Stony Brook, April 2023