LAST TIME:

MAY 3 (1)

THE SCHWARZ-CHRISTOFFEL FORMULA

THM LET $U \subset \mathbb{C}$ BE A POLYGON WITH VERTICES $V_1, V_2, \dots, V_n$ AND INTERNAL ANGLES $\pi\theta_1, \pi\theta_2, \dots, \pi\theta_n$ EACH IN $(0, 2\pi]$.

LET $f: \mathbb{H} \to U$ BE THE CONFORMAL MAP SENDING $P_1, P_2, \dots, P_{n-1}, \infty$ TO THE VERTICES $V_1, V_2, \dots, V_{n-1}, V_n$

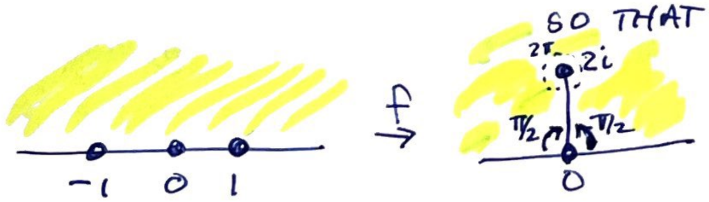
THEN
$$f(z) = f(z_0) + C \int_{z_0}^{z} \prod_{k=1}^{n-1} (\xi - P_k)^{\theta_k - 1}\, d\xi$$
FOR $z_0 \in \mathbb{H} \cup \mathbb{R}$ AND A SUITABLE CONSTANT $C \neq 0$.

AS AN EXAMPLE, LET'S EXPLICITLY FIND THE CONFORMAL MAP TAKING THE UPPER HALF PLANE TO THE UPPER HALF-PLANE SLIT FROM 0 TO $2i$ (REPLACE 2 WITH ANY POSITIVE CONSTANT).

SO THAT $f(-1) = f(1) = 0$

$f(0) = 2$

$f(\infty) = \infty$



THE IMAGE IS A "RECTANGLE" WITH VERTICES AT $0, 2i, 0, \infty$

ANGLES $\pi/2, 2\pi, \pi/2, \pi$

SO BY SCHWARZ-CHRISTOFFEL (USING $z_0 = 1$, $f(z_0) = 0$)

$$f(z) = 0 + C\int_1^z (\xi+1)^{-1/2} \cdot \xi \cdot (\xi - 1)^{-1/2}\, d\xi = C\int_1^z \frac{\xi}{\sqrt{\xi^2-1}}\, d\xi$$

$$f(z) = C\sqrt{z^2-1}$$

WHERE THE BRANCH OF THE $\sqrt{\ }$ IS POSITIVE ON $(1, \infty)$

SINCE $f(0) = 2i$,

THE RESULTING MAP IS

$$f(z) = 2\sqrt{z^2-1}$$

(WITH THE CHOICE OF BRANCH OF SQUARE ROOT)

NOW, ON WITH NEW STUFF

UNLESS OTHERWISE STATED, WE ASSUME ANY TOPOLOGICAL SPACE IS HAUSDORFF, PATH CONNECTED & LOCALLY PATH CONNECTED, AND ALL NEIGHBORHOODS ARE OPEN.

"RECALL" FROM TOPOLOGY THAT A CONTINUOUS MAP

- $f: \tilde{X} \to X$ IS A <u>COVERING MAP</u> ~~IFF~~ EVERY $x \in X$ HAS A NBHD W WHICH IS EVENLY COVERED BY f, IE $f^{-1}(W)$ IS THE DISJOINT UNION OF OPEN SETS $\mathcal{O}_\alpha$ SO THAT $f: \mathcal{O}_\alpha \to W$ IS A HOMEOMORPHISM FOR EACH α. IN THIS CASE, $\tilde{X}$ IS A <u>COVERING SPACE</u> OF X

- IF W IS EVENLY COVERED BY $f: \tilde{X} \to X$, THE <u>FIBER</u> OF x IS $f^{-1}(x)$ AND ITS CARDINALITY DOESNT DEPEND ON x. SINCE X IS CONNECTED, $\mathrm{card}(f^{-1}(x))$ IS CONSTANT AND IS A POSITIVE INTEGER OR $+\infty$. THE <u>DEGREE OF f</u> $= \deg f = \mathrm{card}(f^{-1}(x))$

- LET $f: \tilde{X} \to X$ AND $h: Z \to X$ BE CONTINUOUS. THE MAP $\tilde{h}: Z \to \tilde{X}$ WITH $f \circ \tilde{h} = h$ IS A <u>LIFT</u> OF h (UNDER f)

$$\begin{tikzcd} Z \arrow[r, "\tilde{h}"] \arrow[dr, "h"'] & \tilde{X} \arrow[d, "f"] \\ & X \end{tikzcd}$$

IF TWO LIFTS AGREE AT ONE POINT OF Z, THEY AGREE EVERYWHERE.

DEF A CONTINUOUS SURJECTIVE $f: Y \to X$ HAS THE <u>CURVE LIFTING PROPERTY</u> IF FOR EVERY CURVE $\gamma: [0,1] \to X$ AND EVERY $y_0 \in Y$ SO THAT $f(y_0) = \gamma(0)$, THERE IS A LIFT $\tilde{\gamma}: [0,1] \to Y$ OF γ WITH $\tilde{\gamma}(0) = y_0$

NOTE THAT f A LOCAL HOMEOMORPHISM WITH CURVE LIFTING ~~IMPLIES~~ PROPERTY IMPLIES UNIQUENESS OF LIFTS, BUT $f(z) = z^2$ HAS THE CURVE LIFING PROPERTY BUT NOT UNIQUENESS: EVERY CURVE WITH INITIAL POINT 0 HAS TWO LIFTS UNDER f.

FOR f A LOCAL HOMEOMORPHISM
f A COVERING MAP $\Leftrightarrow$ f HAS CURVE LIFING PROPERTY
~~(LOCAL HOMEO + CURVE LIFTING) ⇒ COVERING MAP~~

DEF/ A COVERING MAP $f: \tilde{X} \to X$ IS <u>UNIVERSAL</u> IF $\tilde{X}$ IS SIMPLY CONNECTED ($\tilde{X}$ IS THE UNIVERSAL COVER OF X). ~~THE~~ SUCH MAPS ARE UNIQUE UP TO ISOMORPHISM.

DEF A <u>DECK TRANSFORMATION</u> OF A COVERING MAP $f: \tilde{X} \to X$ IS A HOMEOMORPHISM $\varphi: \tilde{X} \to \tilde{X}$ THAT PRESERVES FIBERS OF f, IE $f \circ \varphi = f$.
THE COLLECTION OF ALL SUCH φ FORM A GROUP, THE <u>DECK GROUP</u> OF $f: \tilde{X} \to X$.

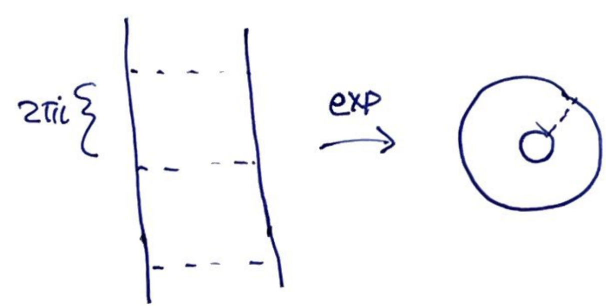
THM: LET G BE THE DECK GROUP OF ~~A COVER $f: \tilde{X}$~~ ~~THE~~ UNIVERSAL COVER $f: \tilde{X} \to X$.

- THE QUOTIENT $\tilde{X}/G$ IS HOMEOMORPHIC TO X, WHERE $\tilde{X}/G$ ~~IS~~ ALL ORBITS $\{\varphi(y) : \varphi \in G\}$ FOR EACH $y \in \tilde{X}$.
- FOR ANY BASE POINT $x_0 \in X$, THE FUNDAMENTAL GROUP $\pi_1(X, x_0) \cong G$.

EXAMPLE: LET A BE THE ANNULUS $A = \{a < |z| < b\}$ WITH $0 < a < b < \infty$. THE UNIVERSAL COVER IS THE STRIP $\tilde{A} = \{ z \mid \ln a < Re(z) < \ln(b) \}$; $\exp(\tilde{A}) = A$ WITH DECK GROUP G GENERATED BY TRANSLATION

$T(z) = z + 2\pi i$. OBSERVE THAT $(\exp \circ T)(z) = \exp(z)$ AND IF $\varphi \in G$ $z \mapsto (\varphi(z) - z)/2\pi i$ IS INTEGER VALUED AND HENCE CONSTANT.

SO EACH $\varphi \in G$ SATISFIES $\varphi = T^{\circ n}$.

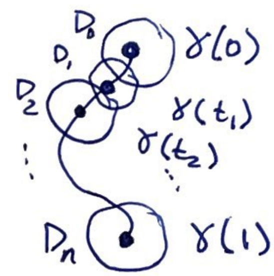


THM: A SURJECTIVE HOLOMORPHIC MAP $f: U \to V$ WITHOUT CRITICAL POINTS IS A COVERING MAP $\iff$ EACH LOCAL BRANCH OF f^{-1} IN V CONTINUES ANALYTICALLY ALONG EVERY CURVE IN V

(BY LOCAL BRANCH OF f^{-1} IN V, WE MEAN A PAIR (g, D) WITH D A DISK IN V, $g(D) \subset U$ AND $f(g(z)) = z$ FOR ALL $z \in D$)

PROOF: SPOZE f IS A COVERING MAP FROM $U \to V$, AND LET (g_0, D_0) BE A LOCAL BRANCH OF f^{-1} IN V, WITH $B_0 = g(D_0)$. LET $\gamma : [0,1] \to V$ BE A CURVE WITH $\gamma(0)$ THE CENTER OF D_0. SINCE f IS A COVERING MAP, IT HAS THE CURVE LIFTING PROPERTY AND THERE IS A PARTITION $0 = t_0 < t_1 < t_2 < \cdots < t_n = 1$ SO THAT $\gamma([t_k, t_{k+1}])$ IS CONTAINED IN AN EVENLY COVERED DISK D_k, $0 \le k < n$.

WE CAN DEFINE (g_k, D_k) INDUCTIVELY:



~~LET $g_k : D_k \to B_k$ BE THE INVERSE~~

LET B_k BE THE CONNECTED COMPONET OF $f^{-1}(D_k)$ THAT CONTAINS $g_{k-1}(D_{k-1} \cap D_k)$, AND DEFINE $g_k : D_k \to B_k$ BE THE INVERSE OF THE BIHOLOMORPHIC RESTRICTION $f : B_k \to D_k$. ON THE INTERSECTION $D_{k-1} \cap D_k$, WE HAVE $f \circ g_{k-1} = f \circ g_k$, AND THE IMAGE OF THE INTERSECTION UNDER g_k AND g_{k-1} ARE CONTAINED IN B_k WHERE f IS INJECTIVE, $g_k = g_{k-1}$ ON THE INTERSECTION, SO (g_0, D_0) CONTINUES ANALYTICALLY ALONG γ.

CONVERSELY, SPOZE EACH LOCAL BRANCH OF f^{-1} CONTINUES ANALYTICALLY ALONG EVERY CURVE IN V. TAKE ANY SUCH CURVE γ AND SOME $p \in f^{-1}(\gamma(0))$. SINCE p IS NOT A CRITICAL POINT, THERE IS A LOCAL BRANCH (g, D) OF f^{-1} WHERE $\gamma(0)$ THE CENTER OF D, WITH $g(\gamma(0)) = p$. SINCE (g, D) CONTINUES ANALYTICALLY ALONG γ, THE LIFT $\tilde{\gamma}$ OF γ STARTS AT p AND $f \circ \tilde{\gamma} = \gamma$. IE, f HAS CURVE LIFTING PROP, AND NO CRITS $\Rightarrow$ f COVER

RECALL THAT AN ASYMPTOTIC VALUE OF $f: U \to V$ IS A $q \in V$ FOR WHICH ~~$f(z) \neq c$ FOR ALL $z \in U$, BUT~~ THERE IS AN ESCAPING CURVE $\gamma: [a,b) \to V$ WITH $\lim_{t \to b} f(\gamma(t)) = q$.

(A CURVE $\gamma: [a,b) \to V$ IS ESCAPING IF FOR EVERY COMPACT $K \subset V$, THERE IS AN $\varepsilon > 0$ SUCH THAT $\gamma(t) \notin K$ FOR $b - \varepsilon < t < b$.)

EXAMPLE: 0 IS AN ASYMPTOTIC VALUE FOR $\exp: \mathbb{C} \to \mathbb{C}$.

THM: LET $f: \Omega \to f(\Omega)$ BE A NONCONSTANT HOLOMORPHIC FUNCTION, $V \subset f(\Omega)$ A DOMAIN CONTAINING NO ASYMPTOTIC VALUES OF f, AND $U \subset \Omega$ A CONNECTED COMPONENT OF $f^{-1}(V)$ CONTAINING NO CRITICAL POINTS OF f. THEN $f: U \to V$ IS A COVERING MAP.

Pf/ SINCE U HAS NO CRITICAL POINTS OF f, $f|_U$ IS A LOCAL HOMEOMORPHISM. THUS THE ONLY OBSTRUCTION TO BEING ABLE TO LIFT ANY CURVE $\gamma(t)$ IN $f(U)$ WOULD BE AN ESCAPING CURVE. ~~BUT SINCE THERE ARE NO~~ ASYMPTOTIC VALUES IN V ~~$f(\gamma), \gamma: [0,1)$~~ $\Rightarrow$ THIS IS NOT POSSIBLE.

COR: GLOBAL INVERSE BRANCHES

LET $f: \Omega \to f(\Omega)$ BE NONCONSTANT, $f \in \mathcal{O}(\Omega)$, $V \subset f(\Omega)$ SIMPLY CONNECTED WITH NO CRITICAL OR ASYMPTOTIC VALUES IN V.

THEN EVERY CONNECTED COMPONENT U OF $f^{-1}(V)$ IS SIMPLY CONNECTED WITH $f: U \to V$ A BIHOLOMORPHISM

IE, THERE ARE HOLOMORPHIC BRANCHES OF f^{-1} IN V.

Pf/ THIS FOLLOWS FROM THE FACT THAT $f: U \to V$ IS A COVERING MAP WITH V SIMPLY CONNECTED. ∎

EXAMPLE:

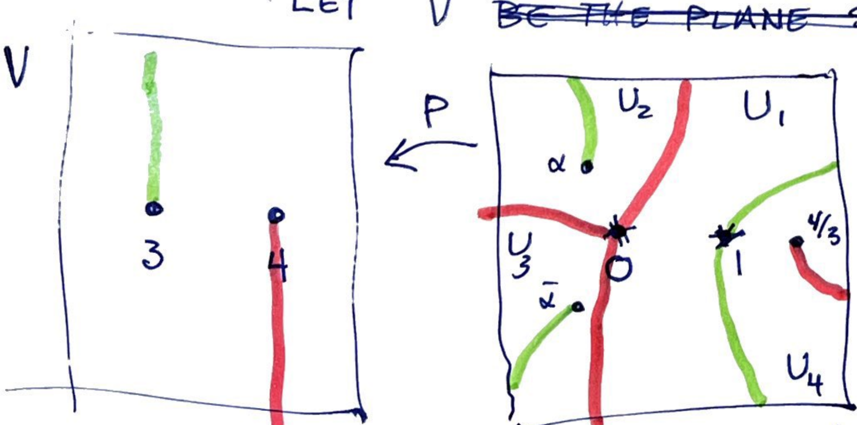
CONSIDER THE POLYNOMIAL $P(z) = 3z^4 - 4z^3 + 4$,

WITH $P'(z) = 12z^3 - 12z^2 = 12z^2(z-1)$.

P HAS CRITICAL ~~VALUES~~ POINTS OF LOCAL DEGREE 2 (AT 0) AND 1 (AT 1)

CRIT VALUES AT $P(0) = 4$, $P(1) = 3$.

LET V ~~BE THE PLANE SLIT~~ A SIMPLY CONNECTED SUBSET OF $\mathbb{C} - \{3, 4\}$.



NOTE THAT

$P(0) = P(4/3) = 4$

$P(1) = P(\alpha) = P(\bar{\alpha}) = 3$.

THERE ARE FOUR HOLO BRANCHES OF P^{-1} MAPPING V ONTO SIMPLY CONNECTED SETS U_1, U_2, U_3, U_4.

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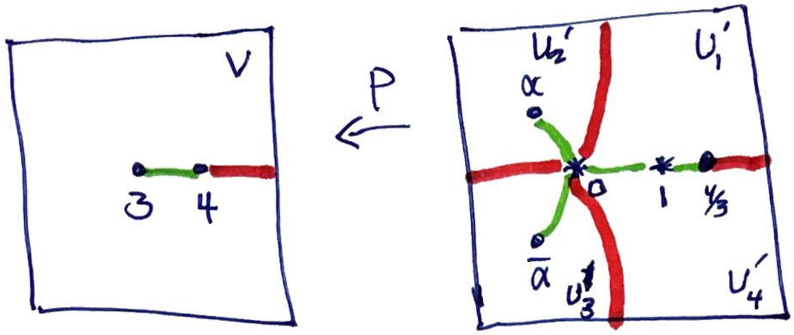
WE CAN SLIT ONCE, AS WELL.

SAME MAP $P: \mathbb{C} \to \mathbb{C}$



~~SINCE 0 IS A ...~~

P IS LOCALLY DEGREE 3 AT 0

ANY PREIMAGE OF A CURVE LANDING AT $4 = P(0)$ MUST CONSIST OF 3 CURVES LANDING AT 0, AND ONE OTHER LANDING AT 4/3.

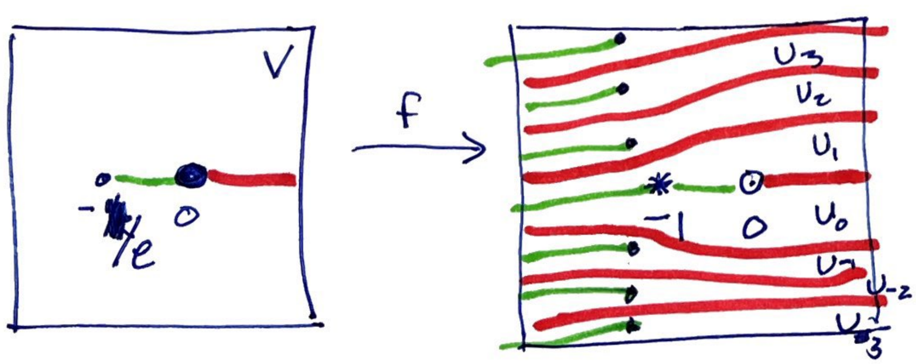
SIMILARLY, SINCE P HAS LOCAL DEGREE 2 AT 1, PREIMAGES OF CURVES LANDING AT $P(1) = 3$ MUST BE TWO CURVES LANDING AT 1 AND TWO OTHERS, EACH LANDING AT α AND $\bar{\alpha}$.

ANOTHER EXAMPLE $\quad f(z) = ze^z$ $\quad$ (BRANCHES OF f^{-1} ARE THE LAMBERT W-FUNCTION)

$f'(z) = (1+z)e^z$, SO THERE IS A CRITICAL POINT OF f AT $z=-1$, AND AN ASYMPTOTIC VALUE AT 0 (SINCE THE LIMIT OF A CURVE γ ESCAPING ~~TO THE LEFT~~ WITH $Re(\gamma) \to -\infty$ IS 0)

NOTE $f(0) = 0$

$f(-1) = -1/e$ LOCAL DEGREE 2.



THERE ARE COUNTABLY MANY DOMAINS U_n ($n \in \mathbb{Z}$) ON WHICH f^{-1} IS HOLOMORPHIC.

NOTE THAT ~~EE~~ THE DOMAINS U_i FOR THE INVERSE CAN "FIT TOGETHER" TO FORM A <u>BRANCHED COVER</u> (OR RAMIFIED COVERING)

~~AS ANE~~

DEF: A CONTINUOUS MAP $f: U \to V$ IS <u>PROPER</u> IF FOR EVERY $K \subset V$ COMPACT, $f^{-1}(K) \subset U$ IS COMPACT

THIS HOLDS $\Leftrightarrow$ FOR EVERY ESCAPING SEQUENCE $\{p_n\}$ IN U (IE $p_n \to \partial U$ IN THE SPHERICAL METRIC), $\{f(p_n)\}$ IS ALSO AN ESCAPING SEQUENCE.

SO $\boxed{\text{PROPER MAPS DO NOT HAVE ASYMPTOTIC VALUES.}}$

DEF: LET U, V BE SIMPLY CONNECTED DOMAINS IN $\hat{\mathbb{C}}$ WITH $p \in U$, $q \in V$, WITH $f: U \to V$ CONTINUOUS. IF THERE ARE HOMEOMORPHISMS $\varphi: (U,p) \to (\mathbb{D},0)$, $\psi: (V,q) \to (\mathbb{D},0)$ AND $m \in \mathbb{N}$ SO THAT $(\psi \circ f \circ \varphi^{-1})(w) = w^m$ FOR $w \in \mathbb{D}$, f IS <u>POWER-LIKE</u>

$$\begin{array}{ccc} (U,p) & \xrightarrow{f} & (V,q) \\ \varphi \downarrow & & \downarrow \psi \\ (\mathbb{D},0) & \xrightarrow{w \mapsto w^m} & (\mathbb{D},0) \end{array}$$

DEF: A CONTINUOUS $f: U \to V$ IS A <u>BRANCHED COVERING</u> IF EVERY $q \in V$ HAS A <u>RAMIFIED NEIGHBORHOOD</u> W_q SO THAT $f^{-1}(W_q)$ IS THE DISJOINT UNION OF OPEN SETS U_K SUCH THAT EACH U_K CONTAINS A UNIQUE PREIMAGE p_K OF q WITH $f(O_K, p_K) \to (W_q, q)$ POWERLIKE.