LAST TIME:

WE SAW THAT THE POISSON INTEGRAL FORMULA GENERALIZES THE MEAN VALUE PROPERTY:

THM (POISSON INTEGRAL FORMULA): LET h BE CONTINUOUS ON $\overline{\mathbb{D}}$, HARMONIC IN $\mathbb{D}$ AND $\zeta = e^{it} \in \mathbb{T}$. THEN

$$h(z) = \frac{1}{2\pi}\int_{\zeta\in\mathbb{T}} P(\zeta, z)\, h(\zeta)\, |dz| = \frac{1}{2\pi}\int_{\zeta\in\mathbb{T}} \mathrm{Re}\left(\frac{\zeta+z}{\zeta-z}\right) h(\zeta)\, |d\zeta| = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{it}-z|^2}\, dt$$

FOR ALL $z \in \mathbb{D}$ ($P(\zeta, z)$ IS THE POISSON KERNEL IN THE DISK)

BUT WE CAN TURN THIS AROUND: GIVEN A CONTINUOUS $h: \mathbb{T} \to \mathbb{C}$, WE HAVE CONSTRUCTED A HARMONIC FUNCTION IN THE INTERIOR $\mathbb{D}$ WHICH EXTENDS CONTINUOUSLY TO $\overline{\mathbb{D}}$.

(THIS IS A SPECIAL CASE OF THE DIRICHLET PROBLEM: GIVEN A FUNCTION DESCRIBING THE BOUNDARY VALUES, FIND A HARMONIC ~~FUNCTION~~ EXTENSION TO THE INTERIOR.)

THE CORRESPONDING $h(z)$ ON $\mathbb{D}$ IS THE POISSON INTEGRAL $\mathcal{P}[h](z)$.

THIS IS A LITTLE MORE GENERAL:

IF $h: \mathbb{T} \to \mathbb{C}$ IS AN L^1 FUNCTION (IE, LEBESGUE INTEGRABLE ON THE CIRCLE) THEN $\mathcal{P}[h]$ IS HARMONIC IN $\mathbb{D}$

~~THEN~~ ~~TO SEE THIS, WRITE e^{it}~~

(DID THE PROOF LAST TIME)

ALSO, FOR $h \in L^1(\mathbb{T})$, THE RADIAL LIMITS EXTEND NICELY AT ANY POINT WHERE $h(\zeta)$ IS CONTINUOUS:

THM (SCHWARZ) IF $h \in L^1(\mathbb{T})$, ~~THEN~~ IS CONTINUOUS AT $\zeta \in \mathbb{T}$

THEN $\lim_{z \to \zeta} \mathcal{P}[h](z) = h(\zeta)$

PF/ ASSUME $h(\zeta) = 0$, (IF NOT, OBSERVE $\mathcal{P}[h - h(\zeta)]$

$= \mathcal{P}[h] - \mathcal{P}[h(\zeta)] = \mathcal{P}[h] - h(\zeta)$)

WRITE $\zeta = e^{it_0}$.

FOR $\varepsilon > 0$, SINCE h IS CONTINUOUS AT ζ, THERE IS AN INTERVAL $I \subset \mathbb{R}$ $I = (t_0 - a_\varepsilon, t_0 + a_\varepsilon)$ WITH $|h(e^{it})| < \varepsilon$ FOR $t \in I$. LET J BE THE COMPLEMENT $[t_0 - \pi, t_0 + \pi] \setminus I$ AND DEFINE

$$h_I(e^{it}) = \begin{cases} h(e^{it}) & t \in I \\ 0 & t \notin I \end{cases} \qquad h_J(e^{it}) = \begin{cases} h(e^{it}) & t \in J \\ 0 & t \notin J \end{cases}$$

SO $h = h_I + h_J$ AND HENCE $\mathcal{P}[h] = \mathcal{P}[h_I] + \mathcal{P}[h_J]$

BUT FOR $z \in \mathbb{D}$,

$$|\mathcal{P}[h_I](z)| \le \frac{1}{2\pi} \int_I P(e^{it}, z) |h(e^{it})| \, dt \le \frac{\varepsilon}{2\pi} \int_I P(e^{it}, z) \, dt \le \varepsilon.$$

BUT ALSO, THERE IS $\delta > 0$ SO THAT $P(e^{it}, z) < \varepsilon$ FOR $z \in \mathbb{D}_\delta(\zeta) \cap \mathbb{D}$ AND $t \in J$.

FOR SUCH A z,

$$|\mathcal{P}[h_J](z)| \le \frac{1}{2\pi} \int_J P(e^{it}, z) |h(e^{it})| \, dt \le \frac{\varepsilon}{2\pi} \int |h(e^{it})| \, dt \le \varepsilon \|h\|_1$$

SO $|\mathcal{P}[h](z)| \le \varepsilon(1 + \|h\|_1)$ AND SO $\lim_{z \to \zeta} \mathcal{P}[h](z) = 0$.

AS A CONSEQUENCE,

COR: EVERY CONTINUOUS MAP $h: \mathbb{T} \to \mathbb{C}$ HAS A UNIQUE CONTINUOUS EXTENSION $H: \overline{\mathbb{D}} \to \mathbb{C}$ WITH H HARMONIC IN $\mathbb{D}$

$$H(z) = \begin{cases} h(z) & \text{FOR } |z|=1 \\ \mathcal{P}[h](z) & \text{FOR } |z|<1 \end{cases}$$

WE JUST NEED TO ESTABLISH THE UNIQUENESS OF H. SPOZE $\tilde{H}$ IS ANOTHER EXTENSION OF h WHICH IS HARMONIC IN $\mathbb{D}$. THEN $\tilde{H} = \mathcal{P}[\tilde{H}|_{\partial\mathbb{D}}] = \mathcal{P}[h] = H$. $\blacksquare$

THIS LETS US COMPLETE THE PROOF OF THE EQUIVALENCE OF ~~THE~~ HARMONIC $\Longleftrightarrow$ LOCAL MEAN VALUE PROPERTY.

Pf/ SPOZE f HAS THE LOCAL MVP ON A DISK $\mathbb{D}_r(p)$ WITH $\overline{\mathbb{D}_r(p)} \subset U$. THEN LET $h = \mathcal{P}[f|_{\mathbb{D}_r(p)}]$, WHICH WILL BE HARMONIC, EXTENDING $f|_{|z-p|=r}$ TO $\overline{\mathbb{D}_r(p)}$.

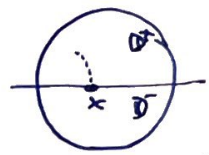
BUT THEN $\psi = f - h$ IS CONTINUOUS ON $\overline{\mathbb{D}_r(p)}$ AND HAS THE LOCAL MVP (SINCE BOTH f AND h DO). BUT SINCE ψ VANISHES IDENTICALLY ON $|z-p|=r$, THE MAXIMUM PRINCIPLE IMPLIES $\psi \equiv 0$ ON $\overline{\mathbb{D}_r(p)}$ (RECALL FROM THE PROOF THIS ONLY DEPENDS ON LOCAL MVP) THUS $h=f$, SO f IS HARMONIC ON $\mathbb{D}_r(p)$.

SINCE $U = \bigcup_{p\in U} \mathbb{D}_{r_p}(p)$ WE HAVE THE RESULT.

THM: HARMONIC SCHWARZ REFLECTION

LET $\mathbb{D}^+ = \{ z \in \mathbb{D} \mid \operatorname{Im}(z) > 0 \}$ WITH $u : \mathbb{D}^+ \to \mathbb{R}$ HARMONIC WITH $u(z) \to 0$ AS $z \to x$ WITH $x \in (-1,1)$

THEN u EXTENDS UNIQUELY TO A HARMONIC FUNCTION ψ ON $\mathbb{D}$



Pf/ LET $\mathbb{D}^- = \{ z \in \mathbb{D} \mid \operatorname{Im}(z) < 0 \}$, AND SET

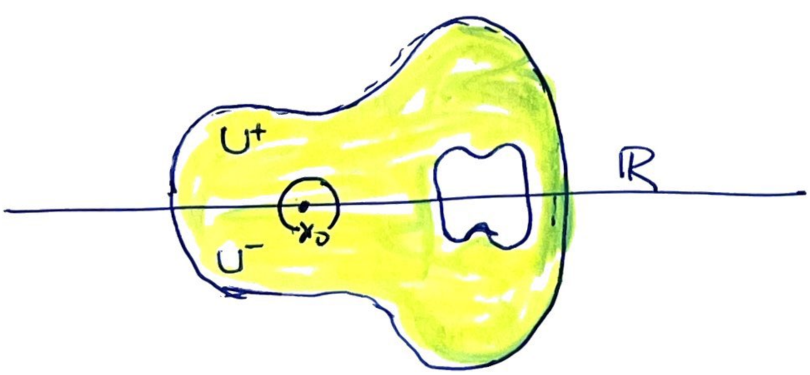
$$\psi(z) = \begin{cases} u(z) & z \in \mathbb{D}^+ \\ 0 & z \in (-1,1) \\ -u(\bar{z}) & z \in \mathbb{D}^- \end{cases}$$

THEN ψ IS CONTINUOUS. IT IS ALSO HARMONIC SINCE IT HAS THE LOCAL MEAN VALUE PROPERTY: FOR ANY DISK AVOIDING $\mathbb{R}$, THIS IS IMMEDIATE SINCE u HARMONIC.

IF $p \in (-1,1)$ AND $\overline{\mathbb{D}_r(p)} \subset \mathbb{D}$,
$$\int_0^{2\pi} \psi(p + re^{it})\, dt$$
$$= \int_0^{\pi} u(p+re^{it})\, dt - \int_{\pi}^{2\pi} u(p+re^{-it})\, dt$$
$$= 0$$

$\blacksquare$

THIS IS A SPECIAL CASE OF THE MORE GENERAL SCHWARZ REFLECTION PRINCIPLE:



SCHWARZ REFLECTION

LET U BE A DOMAIN SYMMETRIC WITH RESPECT TO $\mathbb{R}$, AND $U^+ = \{z \in U \mid \mathrm{Im}(z) > 0\}$. SPOZE f IS ANALYTIC ON U^+ AND $\mathrm{Im}(f(z)) \to 0$ AS $z \in U^+$ TENDS TO $U \cap \mathbb{R}$.

THEN $f(z)$ EXTENDS UNIQUELY TO AN ANALYTIC FUNCTION ON ALL OF U WITH $f(\bar{z}) = \overline{f(z)}$

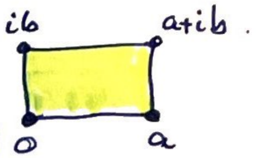
Pf/ WRITE $f(z) = u(z) + iv(z)$ IN U^+. THEN $v(z)$ EXTENDS TO A HARMONIC FUNCTION ON U WITH $v(\bar{z}) = -v(z)$ BY THE PREVIOUS. FIX $x_0 \in U \cap \mathbb{R}$, AND ~~SINCE~~ TAKE A DISK $\mathbb{D}_r(x_0)$ WITH CLOSURE IN U. THEN SINCE v IS HARMONIC IN $\mathbb{D}_r(x_0)$, IT IS THE REAL PART OF A HOLOMORPHIC FUNCTION, WHOSE CONJUGATE MUST COINCIDE WITH f ON $\mathbb{D}_r(x_0) \cap U^+$, SO BY THE IDENTITY THM, f EXTENDS TO $\mathbb{D}_r(x_0)$ ANALYTICALLY.

BUT ALSO $\overline{f(\bar{z})}$ IS ANALYTIC ON $\mathbb{D}_r(x_0)$ AND COINCIDES WITH $f(z)$ ON THE REAL AXIS, SO IT AGREES WITH f IN THE WHOLE DISK $\mathbb{D}_r(x_0)$.

NOW WE CAN EXTEND f TO ALL OF U^- BY $f(z) = \overline{f(\bar{z})}$ WHICH IS ANALYTIC AND COINCIDES WITH THE CONTINUATION OF f.

EXAMPLE: LET R BE A RECTANGLE $(0,a)\times(0,b)$.

DEFINE THE MODULUS OF R $\operatorname{mod}(R) = b/a$.



IF f IS A CONFORMAL MAP WITH $f(R) = R' = (0,a')\times(0,b')$ AND SO THAT EACH VERTEX OF R IS MAPPED TO THE CORRESPONDING VERTEX OF R', THEN $\operatorname{MOD}(R') = b'/a' = b/a$.

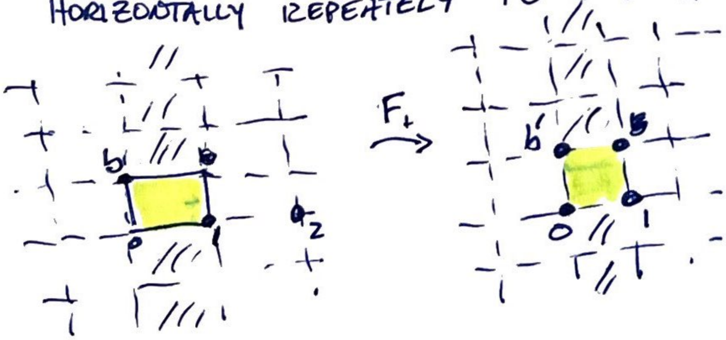
~~EXAMPLE~~ FOR SIMPLICITY, WE CAN ASSUME $a = a' = 1$ (PRE- AND POST-COMPOSE ~~POST-CONJUGATE~~ BY ~~THE~~ AFFINE MAPS) ~~b/a~~.

WE CAN EXTEND f TO THE BOUNDARY OF R: SINCE ∂R AND $\partial R'$ ARE LOCALLY CONNECTED, THE EXTENSION IS CONTINUOUS).

NOW WE CAN EXTEND f BY SCHWARZ REFLECTION TO A MAP $F_1 : (0,1)\times(0,b) \to (0,1)\times(0,b')$

AND REPEAT THE PROCESS TO ~~EX~~ FURTHER EXTEND TO GET

GET A CONFORMAL $F : (0,1)\times\mathbb{R}$ TO ITSELF. THEN EXTEND HORIZONTALLY REPEATEDLY TO GET A CONFORMAL AUTOMORPHISM ~~MAP~~ $\hat{F} : \mathbb{C} \to \mathbb{C}$.



THIS MAP MUST BE AFFINE, IE

$\hat{F}(z) = \alpha z + \beta$.

BUT $\hat{F}(0) = 0$ AND $F(1) = 1$

SO $F(z) = z$. THUS $b = b'$

AND MODULUS IS A CONFORMAL INVARIANT.

ANOTHER APPLICATION IS THE SCHWARZ-CHRISTOFFEL FORMULA, WHICH GIVES AN EXPLICIT FORMULA FOR THE RIEMANN MAP OF A POLYGON (WITH FINITELY MANY SIDES).

THROUGHOUT, FOR $z \in \mathbb{H}$, LET z^α BE THE HOLOMORPHIC BRANCH OF THE POWER FUNCTION, IE

$$z^\alpha = \exp(\alpha \log z) \quad \text{WHERE } \log(z) = \log(re^{it}) = \log r + it$$

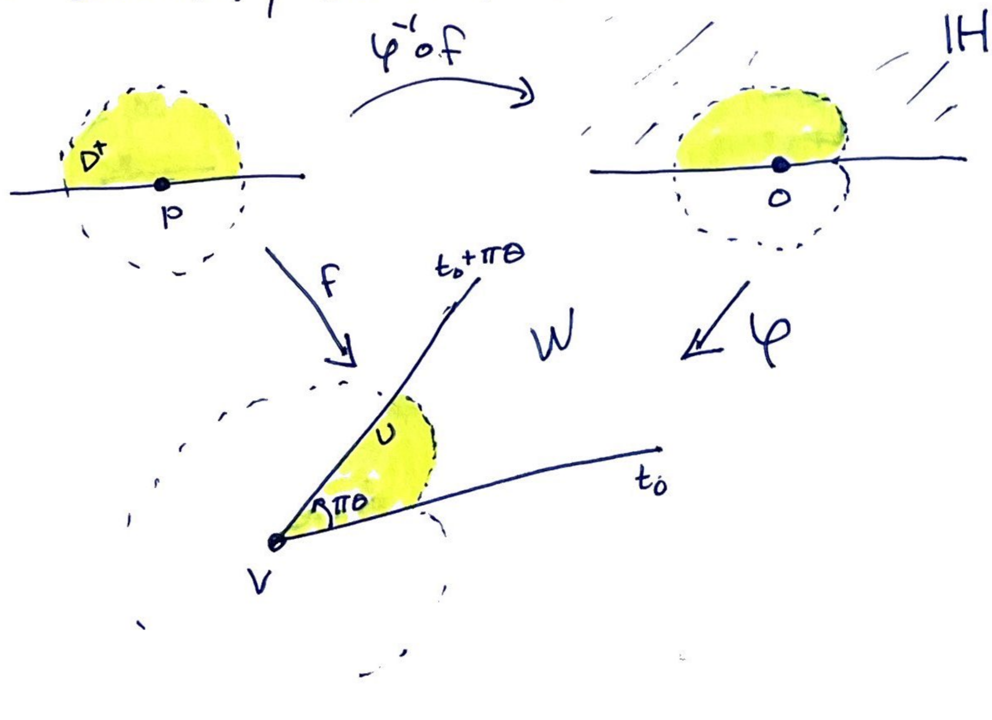
FOR $r > 0$, $0 < t < \pi$.

IN THIS CASE $(z^\alpha)' = \alpha z^{\alpha-1}$.

BEFORE COVERING THE FULL SOLUTION, LETS LOOK AT THE CASE OF A WEDGE OF ANGLE $\pi\theta$ WITH VERTEX V:

$$W = \{ v + re^{it} \mid r > 0,\ t_0 < t < t_0 + \pi\theta \}$$

THEN $\varphi : \mathbb{H} \to W$ GIVEN BY $\varphi(z) = v + e^{it_0} z^\theta$ IS CONFORMAL AND EXTENDS HOMEOMORPHICALLY TO THE BOUNDARY, SENDING $V \to 0$



TAKE $p \in \mathbb{R}$, $\varepsilon > 0$ AND LET $D = \mathbb{D}_\varepsilon(p)$ WITH $D^+ = D \cap \mathbb{H}$.

SPOZE f MAPS D^+ CONFORMALLY ~~MAPS EXTENDS HOMEOMORPHICALLY~~ TO U, THE INTERSECTION OF AN OPEN NBHD OF V WITH W, AND EXTENDS HOMEO. TO ∂D^+, SENDING p TO V, $\partial D^+ \cap \mathbb{R} \to \partial W \cap \partial U$, $\partial D^+ \cap \mathbb{H}$ TO $\partial U \cap W$.

THEN $\varphi^{-1} \circ f : D^+ \to \varphi^{-1}(U)$ EXTENDS HOMEO TO THE BOUNDARY.

BY SHWARTZ REFLECTION, $\varphi^{-1} \circ f$ EXTENDS TO A MAP ON D WITH POSITIVE DERIVATIVE AT P.

THUS WE CAN FIND FIND A NONVANISHING $h \in \mathcal{O}(D)$ WITH $(\varphi^{-1} \circ f)(z) = (z-p)h(z)$ FOR $z \in D$.

THEN $f(z) = V + (z-p)^\theta g(z)$ FOR $z \in D^+$

WITH $g(z)$ A SUITIBLE BRANCH OF $e^{it_0} h^\theta$.

DIFFERENTIATING, WE GET

$$f'(z) = (z-p)^{\theta-1} g_1(z) \quad \text{AND} \quad f''(z) = (z-p)^{\theta-2} g_2(z)$$

WITH $g_1, g_2 \in \mathcal{O}(D)$: $g_1 = \theta g(p) \neq 0$ $\quad g_2 = \theta(\theta-1) g(p)$,

SO $\dfrac{f''(z)}{f'(z)} = \dfrac{1}{z-p} \cdot \dfrac{g_2(z)}{g_1(z)}$ FOR $z \in D^+$. $\left(\text{FOR } |z-p| < \varepsilon, \ \dfrac{f''(z)}{f'(z)} = \dfrac{\theta - 1}{z-p} + \text{ANALYTIC} \right)$

IF $\theta \neq 1$, $\dfrac{f''}{f'}$ ~~IS~~ EXTENDS TO BE MEROMORPHIC IN D WITH A SIMPLE POLE AT P

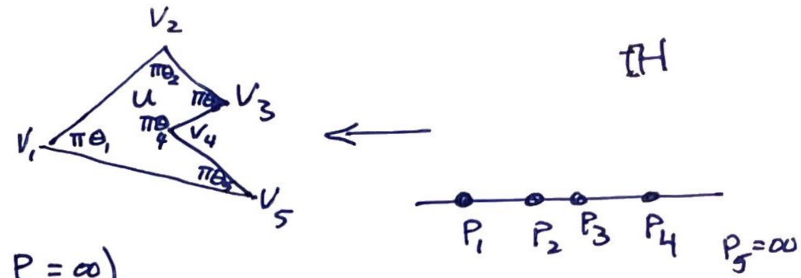
AND IF $\theta = 1$, f''/f' EXTENDS HOLOMORPHICALLY TO D.

A SIMILAR ARGUMENT APPLIES TO $P = \infty$, CHANGING COORDINATES BY $z \to -1/z$
~~HERE WE~~ LETTING $\psi(w) = f(-1/w)$, ψ''/ψ' IS MEROMORPHIC NEAR $w = 0$ WITH AT WORST A SIMPLE POLE, AND

$$\frac{f''(z)}{f'(z)} = 2w + w^2 \frac{\psi''(w)}{\psi'(w)}$$

IS HOLOMORPHIC IN A NEIGHBORHOOD OF ∞ AND TENDS TO 0 AS $z \to \infty$

CONSIDER A POLYGON U WITH VERTICES $V_1, V_2, \dots, V_n$ AND INTERNAL ANGLES $\pi\theta_1, \pi\theta_2, \dots, \pi\theta_n$.



THERE IS A CONFORMAL MAP

$f: \mathbb{H} \to U$ WITH $f(v_j) = P_j$ $(P_n = \infty)$

MAPPING THE ~~SIDES~~ INTERVALS $\overset{I_1}{(-\infty, P_1)}, \overset{I_2}{(P_1, P_2)}, \overset{I_3}{(P_2, P_3)}, \dots, \overset{I_n}{(P_{n-1}, \infty)}$
HOMEOMORPHICALLY TO THE SIDES $[V_n, V_1], [V_1, V_2], \dots, [V_{n-1}, V_n]$

WE CAN REFLECT f ACROSS EACH INTERVAL I_k TO GET A HOLOMORPHIC EXTENSION $F_k : (\mathbb{H} \cup -\mathbb{H} \cup I_k)$ WHICH IS CONFORMAL WITH $F_k' \neq 0$ IN I_k

AND FOR ADJACENT I_j, I_k THE REFLECTED COPIES $F_j(-\mathbb{H})$, $F_k(-\mathbb{H})$ WILL BE ROTATIONS OF EACH OTHER, SO $F_k = \alpha F_j + \beta$, $|\alpha| = 1$, $\alpha, \beta \in \mathbb{C}$

SO $\dfrac{F_k''}{F_k'} = \dfrac{F_j''}{F_j'}$ IN $-\mathbb{H}$,

SO f''/f' EXTENDS TO BE HOLOMORPHIC IN $\mathbb{C} \setminus \{P_1, P_2, \dots, P_n\}$.
WITH SIMPLE POLES OF RESIDUE $\theta_k - 1$ (REMOVABLE IF $\theta_k = \pi$) AT P_k.
AND $\dfrac{f''(z)}{f'(z)} \to 0$ AS $z \to \infty$

THUS $$\frac{f''(z)}{f'(z)} = \sum_{k=1}^{n-1} \frac{\theta_k - 1}{z - P_k}$$

INTEGRATE TO GET $$f'(z) = C \prod_{k=1}^{n-1} (z - P_k)^{\theta_k - 1}$$

FOR A CONSTANT $C \neq 0$.

WE INTEGRATE AGAIN TO GET f EXPLICITLY.

THE SCHWARZ-CHRISTOFFEL FORMULA

THM LET $U \subset \mathbb{C}$ BE A POLYGON WITH VERTICES $V_1, V_2, \dots, V_n$ AND INTERNAL ANGLES $\pi\theta_1, \pi\theta_2, \dots, \pi\theta_n$ EACH IN $(0, 2\pi]$

LET $f : \mathbb{H} \to U$ BE THE CONFORMAL MAP SENDING $P_1, P_2, \dots, P_{n-1}, \infty$ TO THE VERTICES $V_1, V_2, \dots, V_{n-1}, V_n$

THEN
$$f(z) = f(z_0) + C \int_{z_0}^{z} \prod_{k=1}^{n-1} (\xi - P_k)^{\theta_k - 1} \, d\xi$$

FOR $z_0 \in \mathbb{H} \cup \mathbb{R}$ AND A SUITABLE CONSTANT $C \neq 0$.

AS AN EXAMPLE, LET'S EXPLICITLY FIND THE ~~RIEMANN~~ CONFORMAL MAP TAKING THE UPPER HALF PLANE TO THE UPPER HALF-PLANE SLIT FROM 0 TO $2i$ (REPLACE 2 WITH ANY POSITIVE CONSTANT).

SO THAT $f(-1) = f(1) = 0$

$f(0) = 2$

$f(\infty) = \infty$

THE IMAGE IS A "RECTANGLE" WITH VERTICES AT $0, 2i, 0, \infty$

ANGLES $\pi/2, 2\pi, \pi/2, \pi$

SO BY SCHWARZ-CHRISTOFFEL (USING $z_0 = 1$, $f(z_0) = 0$)

$$f(z) = 0 + C\int_1^z (\xi+1)^{-1/2} \cdot \xi \cdot (\xi - 1)^{-1/2} \, d\xi = C\int_1^z \frac{\xi}{\sqrt{\xi^2 - 1}} \, d\xi$$

$$f(z) = C\sqrt{z^2 - 1}$$

WHERE THE BRANCH OF THE $\sqrt{\ }$ IS POSITIVE ON $(1, \infty)$

SINCE $f(0) = 2i$,

THE RESULTING MAP IS

$$f(z) = 2\sqrt{z^2 - 1}$$

(WITH THE CHOICE OF BRANCH OF SQUARE ROOT)