LAST TIME:

A CONTINUOUS FUNCTION $h: U \to \mathbb{C}$ HAS THE MEAN VALUE PROPERTY (MVP)
IF $\overline{\mathbb{D}_r(p)} \subset U \Rightarrow h(p) = \int_0^{2\pi} h(p + re^{it})$. (LOCAL MVP
IF THIS ONLY HOLDS FOR ~~SOME~~ ALL r $0 < r < \delta$ FOR SOME δ)

$\boxed{\text{THM: FOR A CONTINUOUS } h: U \to \mathbb{C}, \text{ WE HAVE } h \text{ HARMONIC} \iff h \text{ HAS THE MEAN VALUE PROP.} \iff h \text{ HAS LOCAL MVP.}}$

IT WILL TAKE SOME TIME TO ESTABLISH THIS FULLY, BUT

h HARMONIC $\Rightarrow$ h HAS MVP $\underset{\text{TRIVIAL}}{\Rightarrow}$ h HAS LOCAL MVP

IS STRAIGHTFOWARD.

Pf/ WE NEED ONLY WORRY ABOUT $h: \mathbb{C} \to \mathbb{R}$ — TO GET THE GENERAL CASE, JUST TAKE REAL & IMAGINARY PARTS.
SO LET $u: U \to \mathbb{R}$ BE HARMONIC WITH $\overline{\mathbb{D}_r(p)} \subset U$.
TAKE $s > r$ SO $\mathbb{D}_s(p) \subset U$ AND THUS FROM LAST TIME HAVING HARMONIC ON A SIMPLY CONNECTED SET, THERE IS $f \in \mathcal{O}(\mathbb{D}_s(p))$ WITH $u = \mathrm{Re}(f)$.
LET $\zeta = p + re^{it}$ FOR $\mathbb{T}_r = \{|z-p| = r\}$ AND BY CAUCHY,

$$f(p) = \frac{1}{2\pi i} \int_{\mathbb{T}_r} \frac{f(\zeta)}{\zeta - p} d\zeta = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(p+re^{it})}{re^{it}} ire^{it} dt = \frac{1}{2\pi} \int_0^{2\pi} f(p+re^{it}) dt$$

SINCE $h(p) = \mathrm{Re}(f(p))$, THE RESULT FOLLOWS.

THM: (MAXIMUM PRINCIPLE) SPOZE $U \subset \mathbb{C}$ IS A BOUNDED DOMAIN WITH h CONTINUOUS IN $\overline{U}$ AND HARMONIC IN U.
THEN $|h(z)| \leq \sup_{\zeta \in \partial U} |h(\zeta)|$ FOR ALL $z \in U$
AND IF h IS REAL VALUED,

$$\inf_{\zeta \in \partial U} h(\zeta) \leq h(z) \leq \sup_{\zeta \in \partial U} h(\zeta) \quad \text{FOR ALL } z \in U$$

IF EQUALITY OCCURS AT $z \in U$, h IS CONSTANT.

SPP/

FIRST, LET'S ASSUME h IS REAL VALUED. ~~AND~~ IF $h(z_0) \geq \sup_{\zeta \in \partial U} h(\zeta)$ WITH $z_0 \in U$, THEN $\sup(h)$ ON $\overline{U}$ IS AT SOME $p \in U$.

LET $E = \{ z \in U \mid h(z) = h(p) \}$.

SINCE h IS HARMONIC, IT HAS THE LOCAL M.V.P., SO FOR ANY $z \in E$, THERE IS A $\delta > 0$ SO THAT FOR ALL $r \in (0, \delta)$

$$\int_0^{2\pi} h(p) - h(z + re^{it})\, dt = 2\pi (h(p) - h(z)) = 0$$

BUT THE INTEGRAND IS CONTINUOUS AND NONNEGATIVE, SO IT IS IDENTICALLY ZERO.

THAT IS, $h(p) = h(z + re^{it})$ (IE, $z \in E$) FOR $z \in \mathbb{D}_r(p)$, IE E IS OPEN SINCE IT CONTAINS A DISK AROUND EVERY POINT.

BUT ALSO, BY THE CONTINUITY OF h, E IS CLOSED IN U, SO $E = U$ AND HENCE h IS CONSTANT.

TO SHOW ~~THE~~ THE OTHER HALF OF THE INEQUALITY, CONSIDER $-h$.

IF h IS COMPLEX VALUED WITH $|h(z_0)| \geq \sup_{\zeta \in \partial U} |h(\zeta)|$ AT SOME $z_0 \in U$, $\sup |h|$ OCCURS AT SOME $p \in U$. IF $h(p) = 0$, THEN h VANISHES ON ALL OF U. OTHERWISE LET $\psi(z) = \frac{h(z)}{h(p)}$, AND ψ IS ALSO HARMONIC.

BUT $|\psi| \leq 1$ WITH $\psi(p) = 1$, SO $\mathrm{Re}(\psi)$ (WHICH IS HARMONIC) TAKES ITS MAXIMUM AT p, AND SO BY THE PREVIOUS, $\mathrm{Re}(\psi) \equiv 1$ THROUGHOUT U.

BUT IF $|\psi| \leq 1$ WITH $\mathrm{Re}(\psi) = 1$, $\psi(z) = 1$ FOR ALL z.

THAT IS $h(z) = h(p)$.

THM: (POISSON INTEGRAL FORMULA) LET h BE CONTINUOUS IN $\overline{\mathbb{D}}$, AND HARMONIC IN $\mathbb{D}$. THEN

$$h(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{it}-z|^2}\, h(e^{it})\,dt \quad \text{FOR } z\in\mathbb{D}.$$

IN OTHER WORDS, NOT ONLY IS THE VALUE AT THE CENTER OF A DISK DETERMINED BY THE BOUNDARY VALUES, THIS HOLDS AT ALL POINTS ~~ON THE BOUNDARY~~ IN THE DISK; $h(z)$ IS THE AVERAGE OF THE VALUES ON THE CIRCLE, WEIGHTED BY THE FACTOR

$$\frac{1-|z|^2}{|\zeta-z|^2} \quad \text{AS } \zeta \text{ RANGES OVER } \partial\mathbb{D}.$$

DEF THIS FACTOR $P(\zeta,z) = \mathrm{Re}\left(\frac{\zeta+z}{\zeta-z}\right) = \frac{1-|z|^2}{|\zeta-z|^2}$ IS THE <u>POISSON KERNEL</u> IN THE UNIT DISK $(\zeta = e^{it})$

PF/ FIX SOME $z\in\mathbb{D}$. $w=\varphi(\zeta) = \dfrac{\zeta-z}{1-\bar{z}\zeta} \in \mathrm{Aut}(\mathbb{D})$

SENDS $z\to 0$. $\quad \zeta = \varphi^{-1}(w) = \dfrac{w+z}{1+\bar{z}w} \in \mathrm{Aut}(\mathbb{D})$.

LET $\tilde{h} = h\circ\varphi^{-1}$.

$\tilde{h}$ IS CONTINUOUS ON $\overline{\mathbb{D}}$, HARMONIC IN $\mathbb{D}$ AND BY MVP,

$$\tilde{h}(0) = \frac{1}{2\pi}\int_0^{2\pi} \tilde{h}(re^{it})\,dt \quad \text{FOR } 0<r<1$$

NOTE $\tilde{h}(re^{it}) \to \tilde{h}(e^{it})$ UNIFORMLY AS $r\to 1$ ON $[0,2\pi]$, SO

$$h(z) = \tilde{h}(0) = \frac{1}{2\pi}\int_0^{2\pi} \tilde{h}(e^{it})\,dt = \frac{1}{2\pi}\int_0^{2\pi} \tilde{h}(\omega)\,|d\omega| = \frac{1}{2\pi}\int_{\mathbb{T}} h(\zeta)\,|\varphi'(\zeta)|\,|d\zeta|$$

FOR $|\zeta|=1$, $|\varphi'(\zeta)| = \dfrac{1-|z|^2}{|1-\bar{z}\zeta|} = \dfrac{1-|z|^2}{|\zeta(\bar{\zeta}-\bar{z})|^2} = \dfrac{1-|z|^2}{|\zeta||\overline{\zeta-z}|^2} = \dfrac{1-|z|^2}{|\zeta-z|^2}$

SO THEN

$$h(z) = \frac{1}{2\pi}\int_{\mathbb{T}} h(\zeta)\frac{1-|z|^2}{|\zeta-z|^2}\,|d\zeta| = \frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|e^{it}-z|^2}\,h(e^{it})\,dt.$$

$\boxtimes$

COR IF $u:\overline{\mathbb{D}}\to\mathbb{R}$ IS CONTINUOUS & HARMONIC ON $\mathbb{D}$. LET

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\,u(e^{it})\,dt, \quad \text{FOR } z\in\mathbb{D}.$$

THEN $f\in\mathcal{O}(\mathbb{D})$ AND $u(z) = \operatorname{Re}(f(z))$

PF/ JUST NOTICE $\operatorname{Re}\left(\dfrac{e^{it}+z}{e^{it}-z}\right) = \operatorname{Re}\left(\dfrac{(e^{it}+z)(e^{-it}-\bar{z})}{|e^{it}-z|^2}\right) = \dfrac{1-|z|^2}{|e^{it}-z|^2}.$

$\boxtimes$

NOTE WE CAN WRITE THE POISSON KERNEL AS

$$P(e^{it}, z) = \operatorname{Re}\left(\frac{e^{it}+z}{e^{it}-z}\right) = \frac{1-|z|^2}{|e^{it}-z|^2}.$$

LET $z = re^{i\theta}$ TO GET

$$P(e^{it}, z) = \frac{1-r^2}{|1-re^{-i(t-\theta)}|^2} = \frac{1-r^2}{1-2r\cos(t-\theta)+r^2}$$

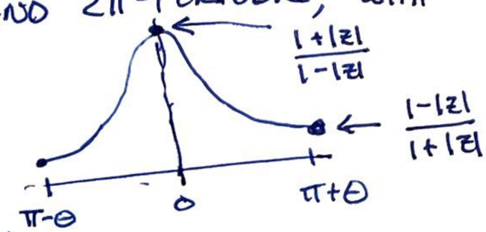
- IT CAN BE USEFUL TO THINK OF THE POISSON KERNEL EITHER AS A FAMILY OF 2π-PERIODIC FUNCTIONS $t \to P_z(e^{it})$ OR AS A FAMILY OF FUNCTIONS $z \to P_t(z)$ ON THE DISK PARAMETERIZED BY $t \in \mathbb{T}$.

- NOTE ALSO THAT $P(e^{it}, z) = P(1, e^{-it}z)$,

  THE FUNCTIONS $z \mapsto P_t(z)$ DIFFERS FROM $z \mapsto P_1(z)$ BY ROTATION.

Thm: FIX $z = |z|e^{i\theta}$ IN $\mathbb{D}$.

(i) $t \mapsto P_z(e^{it}) = P(e^{it}, z)$ IS POSITIVE AND 2π-PERIODIC, WITH GRAPH SYMMETRIC ABOUT $t = \theta$



(ii) $\max_{t\in\mathbb{R}} P_z(e^{it}) = \frac{1+|z|}{1-|z|}$, $\min_{t\in\mathbb{R}} P_z(e^{it}) = \frac{1-|z|}{1+|z|}$

(iii) $\frac{1}{2\pi}\int_0^{2\pi} P(e^{it}, z)\,dt = 1$

THM: FIX $\zeta = e^{it}$ SO $\zeta \in \mathbb{T} = \partial\mathbb{D}$.

(i) THE FUNCTION $z \mapsto P_t(z) = P(\zeta, z)$ IS HARMONIC IN $\mathbb{D}$.

(ii) $\lim\limits_{z\to\zeta} P(\zeta, z) = 0$ UNIFORMLY ON COMPACT SUBSETS OF $\mathbb{T} \setminus \{\zeta\}$

(iii) FOR EVERY $c \in [0, +\infty]$, THERE IS A SEQUENCE $\{z_n\}$ WITH $z_n \to \zeta$ AND $\lim\limits_{n\to\infty} P(\zeta_0, z_n) = c$

Pf/ (i) IS EASY, SINCE $P_t(z) = \operatorname{Re}\left(\frac{e^{it}+z}{e^{it}-z}\right)$, WHICH IS HARMONIC IN $\mathbb{D}$.

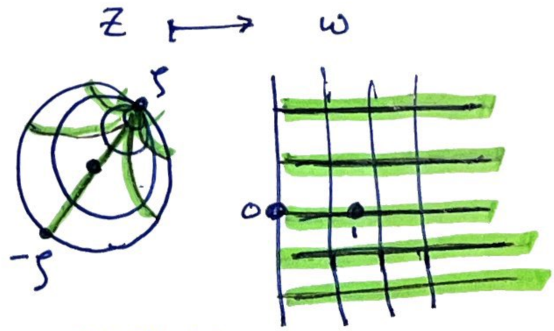
(ii) FOLLOWS FROM OBSERVING THAT AS $z \to \zeta$, $1-|z|^2 \to 0$ BUT $|\zeta - z|^2$ IS UNIFORMLY BOUNDED AWAY FROM 0 ON COMPACT SUBSETS OF $\mathbb{T} - \{\zeta\}$

(iii) SEND $\mathbb{D}$ TO THE RIGHT HALF PLANE VIA $w = \frac{\zeta + z}{\zeta - z}$.

THEN $\zeta \to \infty$, $-\zeta \to 0$, $0 \to 1$

AND HERE, THE POISSON KERNEL

$P(\zeta, z)$ BECOMES $\operatorname{Re}(w)$



IF $z \to \zeta$ ALONG THE PREIMAGE OF A VERTICAL LINE, $\operatorname{Re}(w) = c$, THEN $P(\zeta, z) = c$

HOWEVER, SINCE THE HORIZONTAL LINES PULL BACK TO CIRCULAR ARCS ORTHOGONAL TO $\mathbb{T}$ AND MEETING AT ζ, IF $z \to \zeta$ ALONG SUCH AN ARC, $P(\zeta, z) \to +\infty$.

ANY SEQUENCE IN THE HALF PLANE WITH $\operatorname{Re}(w_n) \to 0$, $|\operatorname{Im}(w_n)| \to +\infty$ GIVES $\{z_n\}$ SO THAT $P(\zeta, z_n) \to 0$

NOTE THAT THE POISSON INTEGRAL FORMULA GIVES US A WAY TO MANUFACTURE A ~~[illegible]~~ HARMONIC FUNCTION IN THE DISK (AND HENCE ANY SIMPLY CONNECTED REGION $U \subset \hat{\mathbb{C}} - \{p, q\}$ VIA THE RIEMANN MAP) FROM AN INTEGRABLE FUNCTION ON THE CIRCLE (RESP ∂U).

LET $L^1(\mathbb{T})$ BE ~~[illegible]~~ THE SPACE OF LEBESGUE INTEGRABLE FUNCTIONS FROM $\mathbb{T} \to \mathbb{C}$ WITH THE NORM

$$\|h\|_1 = \frac{1}{2\pi}\int_0^{2\pi} |h(e^{it})|\, dt$$

DEF: FOR $h \in L^1(\mathbb{T})$, THE POISSON INTEGRAL IS

$$\mathcal{P}[h](z) = \frac{1}{2\pi}\int_{\mathbb{T}} P(\zeta, z)\, h(\zeta)\, |d\zeta|$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{it}-z|^2}\, h(e^{it})\, dt \qquad \text{FOR } z \in \mathbb{D}$$

THM: $h \in L^1(\mathbb{T}) \Rightarrow \mathcal{P}[h]$ IS HARMONIC IN $\mathbb{D}$

- FIRST, IF h IS REAL VALUED, THEN $f \in \mathcal{O}(\mathbb{D})$.

  (WRITE $\frac{e^{it}+z}{e^{it}-z} = 1 + 2\sum_{n=1}^{\infty} e^{-int} z^n$ WHICH CONV. UNIF ON $[0, 2\pi]$ FOR EVERY $z \in \mathbb{D}$

  THEN INTEGRATE TERM BY TERM TO GET THE POWER SERIES FOR f.)

  THUS $\mathcal{P}[h]$ IS HARMONIC IN $\mathbb{D}$.

- IF h IS COMPLEX VALUED, WRITE $\mathcal{P}[h] = \mathcal{P}[\mathrm{Re}(h)] + i\,\mathcal{P}[\mathrm{Im}(h)]$

THM (SCHWARZ) IF $h \in L^1(\mathbb{T})$, ~~THEN~~ IS CONTINUOUS AT $\zeta \in \mathbb{T}$

THEN $\lim_{z \to \zeta} \mathcal{P}[h](z) = h(\zeta)$

Pf/ ASSUME $h(\zeta) = 0$, (IF NOT, OBSERVE $\mathcal{P}[h - h(\zeta)] = \mathcal{P}[h] - \mathcal{P}[h(\zeta)] = \mathcal{P}[h] - h(\zeta)$)

WRITE $\zeta = e^{it_0}$.

FOR $\varepsilon > 0$, SINCE h IS CONTINUOUS AT ζ, THERE IS AN INTERVAL $I \subset \mathbb{R}$ ~~AROUND~~ $I = (t_0 - a_\varepsilon, t_0 + a_\varepsilon)$ WITH $|h(e^{it})| < \varepsilon$ FOR $t \in I$. LET J BE THE COMPLEMENT $[t_0 - \pi, t_0 + \pi] \setminus I$ AND DEFINE

$$h_I(e^{it}) = \begin{cases} h(e^{it}) & t \in I \\ 0 & t \notin I \end{cases} \qquad h_J(e^{it}) = \begin{cases} h(e^{it}) & t \in J \\ 0 & t \notin J \end{cases}$$

SO $h = h_I + h_J$ AND HENCE $\mathcal{P}[h] = \mathcal{P}[h_I] + \mathcal{P}[h_J]$

BUT FOR $z \in \mathbb{D}$,

$$|\mathcal{P}[h_I]| \le \frac{1}{2\pi} \int_I P(e^{it}, z) |h(e^{it})| \, dt \le \frac{\varepsilon}{2\pi} \int_I P(e^{it}, z) \, dt \le \varepsilon.$$

BUT ALSO, THERE IS $\delta > 0$ SO THAT $P(e^{it}, z) < \varepsilon$ FOR $z \in \mathbb{D}_\delta(\zeta) \cap \mathbb{D}$ AND $t \in J$.

FOR SUCH A z,

$$|P[h_J](z)| \le \frac{1}{2\pi} \int_J P(e^{it}, z) |h(e^{it})| \, dt \le \frac{\varepsilon}{2\pi} \int |h(e^{it})| \, dt \le \varepsilon \|h\|_1$$

SO $|P[h](z)| \le \varepsilon (1 + \|h\|_1)$ AND SO $\lim_{z \to \zeta} \mathcal{P}[h](z) = 0$.