Harmonic Functions

The **Laplacian** (or **Laplace operator**) acting on $\mathcal{C}^2$ complex functions is

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}$$

(using the equality of mixed partials $\frac{\partial^2}{\partial x \partial y} = \frac{\partial^2}{\partial y \partial x}$)

Def: A $\mathcal{C}^2$ function $h: U \to \mathbb{C}$ is **harmonic** if the Laplace equation $\Delta h = 0$ for all $z \in U$ (as usual, U is a nonempty & open subset of $\mathbb{C}$)

- Since Δ is linear, the space of all harmonic functions in U is a complex vector space.

Thm: The following are equivalent for a $\mathcal{C}^2$ function $h: U \to \mathbb{C}$:

(i) h is harmonic
(ii) $\mathrm{Re}(h)$ and $\mathrm{Im}(h)$ are harmonic
(iii) $\bar{h}$ is harmonic
(iv) $z \to h(\bar{z})$ is harmonic
(v) $h_z = \frac{1}{2}(h_x - i h_y)$ is holomorphic
(vi) $h_{\bar{z}} = \frac{1}{2}(h_x + i h_y)$ is antiholomorphic.

Pf/ (i) $\Leftrightarrow$ (ii) SINCE $\Delta h = \Delta \mathrm{Re}(h) + i \Delta \mathrm{Im}(h)$

(ii) $\Leftrightarrow$ (iii) SINCE $\Delta \bar{h} = \Delta \mathrm{Re}(h) - i \Delta \mathrm{Im}(h)$

(i) $\Leftrightarrow$ (iv): WRITE $h(z)$ AS $h(x,y)$, SO $h(\bar{z}) = h(x,-y) = \varphi(x,y)$.

THEN $\Delta \varphi(x,y) = \varphi_{xx}(x,y) + \varphi_{yy}(x,y)$

$$= h_{xx}(x,-y) + h_{yy}(x,-y) = \Delta h(x,-y)$$

(i) $\Leftrightarrow$ (v):

$\Delta h = 0 \Leftrightarrow (h_z)_{\bar{z}} = 0$, WHICH IS TRUE BY THE CAUCHY-RIEMANN EQNS $\Leftrightarrow h_z$ IS HOLOMORPHIC.

SIMILARLY $(h_{\bar{z}})_z = 0 \Leftrightarrow h_{\bar{z}}$ IS ANTIHOLOMORPHIC, IE (i) $\Leftrightarrow$ (vi),

EX: $u(x,y) = e^x \cos y$ IS HARMONIC,

SINCE

$$u_z = \tfrac{1}{2}(u_x - i u_y) = \tfrac{1}{2}(e^x \cos y + i e^x \sin y) = \tfrac{1}{2} e^z$$

IS HOLOMORPHIC.

THM: (i) IF f IS HOLOMORPHIC OR ANTIHOLOMORPHIC, THEN f IS HARMONIC

(ii) IF h IS HARMONIC AND f AS ABOVE, THEN $h \circ f$ IS HARMONIC WHEREVER IT IS DEFINED

Pf (i) IS ESSENTIALLY TRIVIAL: SINCE f HOLOMORPHIC, SO IS $f_z = f'$, SO BY (v) ABOVE.

IF f ANTI-HOLOMORPHIC, THEN $\bar{f}$ HOLOMORPHIC AND HENCE HARMONIC, SO f IS HARMONIC.

(ii) IF f HOLOMORPHIC, WE CAN USE THE CHAIN RULE:

$$(h \circ f)_z = (h_z \circ f) f_z + (h_{\bar{z}} \circ f) \underbrace{\bar{f}_z}_{=0 \text{ SINCE } f \text{ IS HOLOMORPHIC}} = (h_z \circ f) f_z$$

, SO $h \circ f$ HARMONIC

FOR ANY $0 \leq a < b \leq +\infty$

IF f IS ANTIHOLOMORPHIC, LET $H(z) = h(\bar{z})$ (WHICH IS HARMONIC)
$h \circ f = H \circ \bar{f}$. SINCE $\bar{f}$ HOLOMORPHIC, $H \circ \bar{f}$ IS HARMONIC
BY THE PREVIOUS PART, SO $h \circ f$ IS.

COR IF $f \in \mathcal{O}(U)$, $\mathrm{Re}(f)$ AND $\mathrm{Im}(f)$ ARE HARMONIC

EXAMPLES:

- $h(z) = \mathrm{Im}\left(\frac{1}{z^2}\right)$ IS HARMONIC FOR $z \neq 0$ ($u(x,y) = \frac{-2xy}{(x^2+y^2)^2}$

  SINCE h IS IDENTICALLY 0 FOR $x=0$ OR $y=0$
  $h_{xx}(0) = h_{yy}(0) = 0$.
  SO $\Delta h \equiv 0$ EVERYWHERE
  BUT h IS NOT HARMONIC ON $\mathbb{C}$ (ONLY $\mathbb{C}^*$)
  SINCE h IS NOT CONTINUOUS AT 0.

- LET $u(z) = \log|z|$ FOR $z \neq 0$:

  $$\Delta u = 4 \frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} \log|z| = 2 \frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} \log(z\bar{z}) = 2 \frac{\partial}{\partial z}\left(\frac{1}{\bar{z}}\right) = 0$$

  SO u IS HARMONIC WHENEVER $z \neq 0$

  THUS IF $f \in \mathcal{O}(U)$, $\log|f|$ IS HARMONIC
  WHENEVER $f(z) \neq 0$

- **BUT** OBSERVE $u(z) = \log|z|$ ($z \in \mathbb{C}^*$)
  IS NOT $\mathrm{Re}(f)$ FOR $f \in \mathcal{O}(U)$ WITH $U = \{z \mid a < |z| < b\}$
  FOR ANY $0 \leq a < b \leq +\infty$

SPOZE ~~SUCH~~ $u = \mathrm{Re}(f)$ FOR SOME $f \in \mathcal{O}(U)$.

THEN $u = \frac{1}{2}(f + \bar{f})$ ON U, SO $u_z = \frac{1}{2}(f_z + \bar{f}_z) = \frac{1}{2} f'$

AND $\int_{|z|=r} u_z \, dz = 0$ FOR ANY $r \in (a,b)$.

BUT $\int_{|z|=r} u_z \, dz = \frac{1}{2} \int_{|z|=r} \frac{dz}{z} = \pi i \neq 0$.

HOWEVER, EVERY REAL VALUED HARMONIC FUNCTION IS LOCALLY THE REAL PART OF A HOLOMORPHIC FUNCTION:

THM: LET $U \subset \mathbb{C}$ BE A SIMPLY-CONNECTED DOMAIN, WITH $u: U \to \mathbb{R}$ BE HARMONIC. THEN THERE IS $f \in \mathcal{O}(U)$ WITH $u = \mathrm{Re}(f)$.
f IS A PRIMITIVE OF $2u_z$ AND f IS UNIQUE UP TO ADDITION OF A PURELY IMAGINARY CONSTANT

Pf/ $2u_z \in \mathcal{O}(U)$, SO IT HAS A PRIMITIVE $g \in \mathcal{O}(U)$

SINCE $(g - 2u)_z = g' - 2u_z = 0$, $g - 2u$ IS ANTI-HOLOMORPHIC

THUS $\overline{g - 2u}$ ~~IS HOLO~~ $= \bar{g} - 2u$ IS HOLOMORPHIC.

SINCE REAL

$g \in \mathcal{O}(U)$, SO $g + \bar{g} - 2u \in \mathcal{O}(U)$. BUT SINCE IT IS REAL VALUED BY THE OPEN MAPPING THM, IT IS A CONSTANT $2c$.

THUS $u = \mathrm{Re}(g - c)$. ~~LET $f = g - c$~~

IF $\tilde{f} \in \mathcal{O}(U)$ WITH $\mathrm{Re}(\tilde{f}) = u$, THEN $\mathrm{Re}((g-c) - \tilde{f}) = 0$, SO $\tilde{f} = Ki$, SOME $K \in \mathbb{R}$

AS A CONSEQUENCE, SINCE EVERY REAL VALUED HARMONIC FUNCTION IS THE REAL PART OF A HOLOMORPHIC FUNCTION AND HENCE INFINITELY DIFFERENTIABLE,

COR EVERY HARMONIC FUNCTION IS C^∞

ALSO

COR (HARMONIC IDENTITY THM): LET $U \subset \mathbb{C}$ BE A DOMAIN WITH $h_1, h_2 : U \to \mathbb{C}$ HARMONIC. IF $h_1 = h_2$ ON SOME NONEMPTY OPEN SUBSET OF U, THEN $h_1 = h_2$ ON ALL OF U

PF/ SINCE WE CAN CONSIDER $\mathrm{Re}(h_1 - h_2)$ AND $\mathrm{Im}(h_1 - h_2)$ SEPARATELY, WE NEED ONLY LOOK AT $h = h_1$ REALVALUED, WITH $h_2 = 0$.

SO LET $V \neq \emptyset$ BE THE MAXIMAL OPEN SUBSET WHERE $h = 0$. IF $V \neq U$, LET $p \in \partial V \cap U$ AND CHOOSE $\mathbb{D}_r(p) \subset U$

BY THE THM, THERE IS $f \in \mathbb{D}_r(p)$ WITH $h = \mathrm{Re}(f)$. BUT f IS PURELY IMAGINARY ON $\mathbb{D}_r(p) \cap V$, SO IT IS CONSTANT ON THIS PART OF $\mathbb{D}_r(p) \cap V$, BUT THEN BY THE IDENTITY THM, $f \in \mathcal{O}(\mathbb{D}_r(p))$ IS CONSTANT. THUS $h = \mathrm{Re}(f)$ VANISHES ON V AND ALSO $\mathbb{D}_r(p)$, SO V WASN'T MAXIMAL. THUS $V = U$ AND h VANISHES ON ALL OF U

NOTE UNLIKE THE HOLOMORPHIC ANALOGUE, MERELY AGREEING ON A SET WITH ACCUMULATION POINTS IS INSUFFICIENT. CONSIDER, FOR EXAMPLE, $f(z) = \mathrm{Re}(z)$, WHICH IS 0 ON THE IMAGINARY AXIS, BUT NOT IDENTICALLY 0.

A VERSION FOR AN ANNULUS (WHICH GENERALIZES TO MULTIPLY-CONNECTED DOMAINS):

THM LET $U = \{ z \in \mathbb{C} \mid a < |z-p| < b \}$, WITH $0 \leq a < b \leq \infty$

FOR ANY HARMONIC $u : U \to \mathbb{R}$, THERE IS $f \in \mathcal{O}(U)$ AND $\alpha \in \mathbb{R}$ SO THAT

$$u(z) = \mathrm{Re}(f(z)) + \alpha \log|z-p| \quad \text{FOR ALL } z \in U$$

α IS UNIQUE, BUT f IS DETERMINED ONLY UP TO A PURELY IMAGINARY CONSTANT. (α IS THE PERIOD OF u)

THE ARGUMENT IS SIMILAR TO THE EARLIER PROOF FOR U SIMPLY CONNECTED. FIX $a < r < b$, LET

$$\alpha = \frac{1}{\pi i} \int_{\partial \Delta_r(p)} u_z \, dz$$

SINCE u IS HARMONIC, $u_z \in \mathcal{O}(U)$

SO α IS INDEPENDENT OF r.

BUT $\alpha = \frac{1}{2\pi i}\int_{\Pi_r(p)} (u_x - i u_y)(dx + i dy)$

SO $\operatorname{Im}(\alpha) = \frac{-1}{2\pi}\int_{\Pi_r(p)} (u_x dx + u_y dy) = \frac{-1}{2\pi}\int_{T_r(p)} du = 0$ SINCE $T_r(p)$ IS CLOSED.

SO $\alpha \in \mathbb{R}$.

NOW LOOK AT $2u_z - \frac{\alpha}{z-p} \in \mathcal{O}(U)$, AND LET γ BE CLOSED CURVE IN U WITH WINDING NUMBER $W(\gamma, p) = n$, SO THE CYCLE $\gamma - n\Pi_p(p)$ IS NULL-HOMOLOGOUS IN U.

$$\int_\gamma \left(2u_z - \frac{\alpha}{z-p}\right)dz = n\int_{\Pi_r(p)}\left(2u_z - \frac{\alpha}{z-p}\right)dz$$
$$= 2n\int_{\Pi_r(p)} u_z dz - n\alpha\int_{\Pi_r(p)}\frac{dz}{z-p}$$
$$= 2n\cdot \pi i\alpha - n\alpha\cdot 2\pi i = 0$$

THUS $2u_z - \frac{\alpha}{z-p}$ HAS A PRIMITIVE $g \in \mathcal{O}(U)$. COMPUTING $\frac{\partial}{\partial z}$ OF THE PRIMITIVE:

$$\frac{\partial}{\partial z}\left(g - 2u + 2\alpha\log|z-p|\right) = \frac{\partial}{\partial z}\left(g - 2u + \alpha\log\left((z-p)(\bar z - \bar p)\right)\right)$$
$$= g' - 2u_z + \frac{\alpha}{z-p} = 0$$

SINCE $\frac{\partial}{\partial z} = 0$, IS ANTIHOLO, SO CONJUGATE IS HOLOMORPHIC

AND

$$g + \bar g - 2u + 2\alpha\log|z-p|$$
$$= 2\left(\operatorname{Re}(g) - u + \alpha\log|z-p|\right) \in \mathcal{O}(U)$$

BUT SINCE IT IS REAL-VALUED, IT IS A CONSTANT $2c$.

THUS $f = g - c$ IS HOLOMORPIC ON U

WITH $\mathrm{Re}(f) = u - \alpha \log|z-p|$. AS BEFORE,

IF

$$\mathrm{Re}(f) + \alpha \log|z-p| = \mathrm{Re}(\tilde{f}) + \tilde{\alpha} \log|z-p|$$

THEN $\mathrm{Re}(\tilde{f} - f) = (\alpha - \tilde{\alpha}) \log|z-p|$ FOR $z \in U$

BUT THEN $\alpha = \tilde{\alpha}$, SO $f, \tilde{f}$ DIFFER BY A PURELY IMAGINARY CONSTANT.

THM (REMOVABLE SINGULARITIES) A BOUNDED HARMONIC FUNCTION IN $\mathbb{D}_r^*(p)$ EXTENDS TO A HARMONIC FUNCTION ON ALL OF $\mathbb{D}_r(p)$

PF/ IT IS SUFFICIENT TO CONSIDER $u : \mathbb{D}_r^*(p) \to \mathbb{R}$, BOUNDED & HARMONIC. FROM THE THM, THERE IS $f \in \mathcal{O}(\mathbb{D}_r^*(p))$ AND $\alpha \in \mathbb{R}$ SO THAT $\mathrm{Re}(f(z)) = u(z) - \alpha \log|z-p|$ FOR $z \in \mathbb{D}_r^*(p)$

IF p IS A REMOVABLE SINGULARITY,

$\alpha \log|z-p| = u(z) - \mathrm{Re}(z)$ IS BOUNDED AS $z \to p$, SO $\alpha = 0$,

AND $\mathrm{Re}(f)$ IS THE EXTENSION OF u TO $\mathbb{D}_r(p)$.

SPOZE p IS NOT REMOVABLE, SO IT MUST BE A POLE OR AN ESSENTIAL SINGULARITY.

DEF A CONTINUOUS $h : U \to \mathbb{C}$ HAS THE MEAN VALUE PROPERTY IF

$$h(p) = \frac{1}{2\pi}\int_0^{2\pi} h(p+re^{it})\,dt$$

FOR ANY $\overline{\mathbb{D}_r(p)} \subset U$. h HAS THE LOCAL MEAN VALUE PROPERTY IF ~~THERE IS~~ FOR EACH $p \in U$, THERE IS $\delta > 0$ SO THAT THE ABOVE HOLDS FOR ALL $0 < r < \delta$

~~WHILE~~ (WHILE THE LOCAL PROPERTY IS WEAKER, WE HAVE

THM: LET $h : U \to \mathbb{C}$ ARE EQUIVALENT

(i) h IS HARMONIC

(ii) h HAS THE MEAN VALUE PROPERTY

(iii) h HAS THE LOCAL MEAN VALUE PROPERTY

THIS WILL TAKE SOME EFFORT TO ESTABLISH, BUT $(i) \Rightarrow (ii) \Rightarrow (iii)$ IS STRAIGHTFORWARD:

PF/ TAKE $u : U \to \mathbb{R}$ HARMONIC, WITH $\overline{\mathbb{D}_r(p)} \subset U$. CHOOSE $s > r$ SO $\mathbb{D}_s(p) \subset U$, AND HENCE THERE IS $f \in \mathcal{O}(\mathbb{D}_s(p))$ WITH $u = \mathrm{Re}(f)$

NOW APPLY THE CAUCHY INTEGRAL FORMULA OVER THE CIRCLE $p + re^{it} = \zeta$

$$f(p) = \frac{1}{2\pi i}\int_{\mathbb{T}_r(p)} \frac{f(\zeta)}{\zeta - p}\,d\zeta = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(p+re^{it})}{re^{it}}\, ire^{it}dt = \frac{1}{2\pi}\int_0^{2\pi} f(p+re^{it})\,dt$$

SO $\mathrm{Re}(f(p)) = \mathrm{Re}\left(\frac{1}{2\pi}\int_0^{2\pi} f(p+re^{it})\,dt\right)$

THM (MAXIMUM PRINCIPLE FOR HARMONIC)

LET $U \subset \mathbb{C}$ BE A BOUNDED DOMAIN, WITH h CONTINUOUS ON $\overline{U}$ HARMONIC IN U

(a) IF $h : U \to \mathbb{R}$, $\inf_{\zeta \in \partial U} |h(\zeta)| \leq h(z) \leq \sup_{\zeta \in \partial U} h(\zeta)$ FOR $z \in U$

(b) IN ANY CASE

$$|h(z)| \leq \sup_{\zeta \in \partial U} |h(\zeta)| \quad \text{FOR } z \in U$$

IF EQUALITY OCCURS AT $z \in U$, THEN h IS CONSTANT.

IN FACT, h ONLY NEEDS THE LOCAL MEAN VALUE PROP.

A MORE GENERAL VERSION:

SPOZE h HARMONIC IN A DOMAIN $U \subset \mathbb{C}$

IF $\limsup_{n \to \infty} |h(z_n)| \leq M$ FOR EVERY ESCAPING SEQUENCE $\{z_n\}$ (IE $\lim_{n \to \infty} \operatorname{dist}(z_n, \partial U) \to 0$)

THEN $|h| \leq M$ THROUGHOUT U

(IF $h : U \to \mathbb{R}$, AND $\limsup_{n} h(z_n) \leq M$ FOR EVERY ESC. $\{z_n\}$

THEN $|h| \leq M$ IN U)