LAST TIME:

THM (CARATHÉODORY): A RIEMANN MAP $\mathbb{D} \to U$ EXTENDS TO A CONTINUOUS MAP $\overline{\mathbb{D}} \to \overline{U}$ $\Leftrightarrow$ ∂U IS LOCALLY CONNECTED

COR: THIS EXTENSION MAP IS A HOMEOMORPHISM $\Leftrightarrow$ ∂U IS A JORDAN CURVE

(PROOF IS THE MAIN GOAL OF TODAY)

RECALL THAT A COMPACT SET $X \subset \hat{\mathbb{C}}$ IS LOCALLY CONNECTED

- IF FOR EVERY $p \in X$ AND EVERY OPEN NBHD Ω OF p, THERE IS AN OPEN NEIGHBORHOOD $V \subset \Omega$ OF p WITH $V \cap X$ CONNECTED

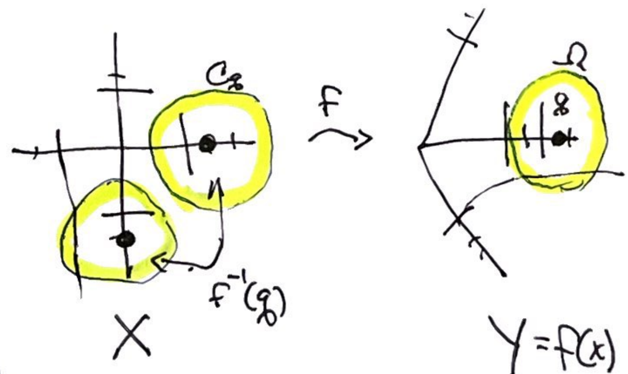
$\Leftrightarrow$ • FOR EVERY $\varepsilon > 0$ THERE IS $\delta > 0$ SO THAT IF $p, q \in X$ WITH $\mathrm{dist}_\sigma(p,q) < \delta$, THEN p & q ARE IN A CONNECTED SET E WITH $\mathrm{diam}_\sigma(E) \leq \varepsilon$.

$\Leftrightarrow$ FOR EVERY OPEN SET $\Omega \subset \hat{\mathbb{C}}$, THE CONNECTED COMPONENTS OF $\Omega \cap X$ ARE RELATIVELY OPEN IN X.

AS A CONSEQUENCE:

LEMMA: LET $X \subset \hat{\mathbb{C}}$ BE COMPACT AND LOCALLY CONNECTED. IF $f: X \to \hat{\mathbb{C}}$ IS CONTINUOUS, THEN $f(X)$ IS LOCALLY CONNECTED.

LET $Y = f(X)$, $\Omega \subset \hat{\mathbb{C}}$ AN OPEN SET CONTAINING A CONNECTED COMPONENT H OF $\Omega \cap Y$.



FOR EACH $q \in H$, LET C_q BE A CONNECTED COMPONENT OF $f^{-1}(\Omega \cap Y)$ MEETING $f^{-1}(q)$

$f^{-1}(\Omega \cap Y)$ IS RELATIVELY OPEN IN X, SO THE COMPONENT C_q IS RELATIVELY OPEN IN X, HENCE $X \setminus C_q$ IS COMPACT.

HENCE $f(X \setminus C_q)$ IS COMPACT,

SO THE COMPLEMENT $V_q = Y \setminus f(X - C_q)$ IS REL OPEN IN Y

WITH $V_q \subset f(C_q)$

BUT ALSO, $f(C_q) \subset (\Omega \cap Y)$ IS CONNECTED AND CONTAINS q_0

SO $f(C_q) \subset H$

IE $V_q \subset H$.

HENCE $H = \bigcup_{q \in H} V_q$ IS RELATIVELY OPEN IN Y

IE Y IS LOCALLY CONNECTED. $\boxtimes$

NOW CONSIDER OPEN CURVES $\gamma : [a,b) \to \hat{\mathbb{C}}$.

THE (SPHERICAL) LENGTH OF SUCH A CURVE IS THE LIMIT OF LENGTHS OF $\gamma(\,[a, b-\varepsilon]\,)$ AS $\varepsilon \to 0$

WHICH COULD BE FINITE OR $+\infty$,

$$\text{length}_\sigma(\gamma) = \lim_{\varepsilon \to 0} \int_a^{b-\varepsilon} \frac{2|\gamma'(t)|}{1+|\gamma(t)|^2}\,dt = \int_a^b \frac{2|\gamma'(t)|}{1+|\gamma(t)|^2}\,dt$$

AS LONG AS γ IS PIECEWISE C^1

IF $\lim_{t \to b} \gamma(t) = p \in \hat{\mathbb{C}}$, WE WILL SAY THE OPEN CURVE γ <u>LANDS AT p</u>

LEMMA: IF $\gamma : [a,b) \to \hat{\mathbb{C}}$ HAS FINITE LENGTH, THEN THE CURVE LANDS AT SOME $p \in \hat{\mathbb{C}}$, IE $\lim_{t \to b} \gamma(t)$ EXISTS

Pf/ LET E BE THE SET OF ACCUMULATION POINTS OF $\gamma(t)$ AS $t \to b$, IE $E = \bigcap_{a<t<b} \overline{\gamma((t,b))}$

$E \neq \emptyset$ AND E IS COMPACT, SINCE IT IS A NESTED INTERSECTION OF NONEMPTY COMPACT SETS.

IF $p, q \in E$ WITH $p \neq q$, WE CAN CHOOSE SUBSEQUENCES t_n, s_n WITH $\gamma(t_n) \to p$, $\gamma(s_n) \to q$

AND $t_n < s_n < t_{n+1}$ WITH $t_n \to b$, $s_n \to b$.

FOR n SUFF. LARGE WE CAN ENSURE

$$\text{dist}_\sigma(\gamma(t_n), \gamma(s_n)) \geq \tfrac{1}{2}\,\text{dist}_\sigma(p,q) > 0$$

BUT $\text{length}_\sigma(\gamma) \geq \text{length}_\sigma(\gamma([a,s_n])) \geq \sum_{k=1}^{n} \text{dist}_\sigma(\gamma(t_k), \gamma(s_k))$

BUT IF $p \neq q$, THE SUM DIVERGES; SO $\text{length}_\sigma(\gamma) = +\infty$,

$\therefore$ $p = q$

WE CAN DEFINE A DOUBLY-OPEN CURVE $\gamma((a,b))$ AS THE UNION OF TWO SUCH CURVES.

DEF: LET $U \subsetneq \hat{\mathbb{C}}$ BE A SIMPLY CONNECTED DOMAIN.
A <u>CROSS-CUT</u> IN U IS A CURVE $\gamma:(a,b) \to U$ WITH γ INJECTIVE, C^1 AND SUCH THAT $\lim_{t\to a} \gamma(t) = p \in \partial U$, $\lim_{t\to b} \gamma(t) = q \in \partial U$

LEMMA: IF $U \subsetneq \hat{\mathbb{C}}$ IS A SIMPLY CONNECTED DOMAIN WITH A CROSS-CUT $\gamma:(a,b) \to U$.
THEN $U \setminus \{\gamma\}$ HAS EXACTLY TWO COMPONENTS U_1, U_2
AND $\partial U_1 \cap U = \partial U_2 \cap U = \{\gamma\}$

Pf/ BY RIEMANN MAPPING THM, $U = \mathbb{C}$ OR $U \overset{\cong}{=} \mathbb{D}$, CONFORMALLY.
BUT BOTH ARE HOMEOMORPHIC TO $\mathbb{C}$ BY SOME φ.
THUS $\eta = \varphi \circ \gamma : (a,b) \to \mathbb{C}$ TENDS TO ∞ IN BOTH DIRECTIONS,
SO $\bar{\eta} : [a,b] \to \hat{\mathbb{C}}$ HAS $\bar{\eta}(a) = \bar{\eta}(b) = \infty$ AND $\{\bar{\eta}\}$ IS A JORDAN CURVE.

COR: LET $f: U \to V$ BE A CONFORMAL MAP BETWEEN SIMPLY CONNECTED DOMAINS U & V, WITH γ A CROSS-CUT IN U WITH IMAGE $\{f \circ \gamma\}$ OF FINITE LENGTH.
THEN η IS A CROSS-CUT IN V
AND f MAPS EACH CONN. COMPONENT OF $U \setminus \{\gamma\}$ TO A CONN. COMP. OF $V \setminus \{\eta\}$

Pf/ • FIRST, SINCE η IS C^1 AND INJECTIVE, AND BOTH ENDS LAND IN ∂V (SINCE $\eta = f\circ\gamma$ HAS FINITE LENGTH AND f A HOMEOMORPHISM) SO η IS A CROSSCUT IN V.

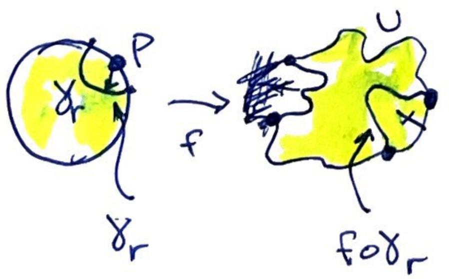
• NOW, LET U_1 BE A COMPONENT OF $U \setminus \{\gamma\}$. $f(U_1)$ IS A CONNECTED SUBSET OF $V \setminus \{\eta\}$, SO $f(U_1) \subset V_1$ WITH V_1 A CONNECTED COMPONENT OF $V \setminus \{\eta\}$. APPLYING THE SAME ARGUMENT TO f^{-1}, WE SEE $f^{-1}(V_1) \subset U_1$ AND SO $f(U_1) = V_1$ ∎

NOW TO PROVE THE THEOREM:

$\Rightarrow$ IS EASY: LET $f: \mathbb{D} \to U$ BE A RIEMANN MAP, AND SUPPOSE IT EXTENDS CONTINUOUSLY TO $\bar{f}: \bar{\mathbb{D}} \to \bar{U}$. SINCE $\bar{f}$ IS CONTINUOUS ~~A HOMEOMORPHISM,~~ $\bar{f}(\partial\mathbb{D}) = \partial U$. BUT SINCE $\partial\mathbb{D}$ IS LOCALLY CONNECTED, SO IS ∂U BY THE LEMMA.

$\Leftarrow$ IS MORE WORK. SPOZE ∂U IS LOCALLY CONNECTED. POST-COMPOSE f WITH A MÖBIUS MAP SO WE CAN ASSUME $f(0) = \infty$. THIS ENABLES US TO USE EITHER THE EUCLIDEAN OR SPHERICAL METRIC, SINCE $f(\mathbb{D}_{1/2}(0))$ IS A NBHD OF ∞, AND HENCE THE IMAGE OF THE ANNULUS $1/2 < |z| < 1$ IS BOUNDED.

LET $p = e^{i\theta}$ BE A POINT ON THE UNIT CIRCLE, AND $\gamma_r(t) = p + re^{it}$, $a_r < t < b_r$ PARAMETERIZE THE OPEN ARC INSIDE $\mathbb{D}$ OF RADIUS r AROUND p.



LET L_r BE THE LENGTH OF $f\circ\gamma_r$ SO

$$(L_r)^2 = \left(\int_{\gamma_r} |f'(z)|\,|dz|\right)^2 \leq \left(\int_{\gamma_r} |dz|\right)\left(\int_{\gamma_r} |f'(z)|^2\,|dz|\right)$$

SO

SINCE γ_r IS LESS THAN HALF A CIRCLE.

$$(L_r)^2 \leq (\pi r)\left(\int_{a_r}^{b_r} |f'(p+re^{it})|^2 r\,dt\right)$$

~~BUT CALLED~~ DIVIDING BY r AND INTEGRATING, WE HAVE

$$\int_0^{1/2} \frac{(L_r)^2}{r}\,dr \leq \pi \int_0^{1/2}\int_{a_r}^{b_r} |f'(p+re^{it})|^2 r\,dt$$

$$= \pi \iint_{\mathbb{D}\cap\mathbb{D}_{1/2}(p)} |f'(z)|^2\,dx\,dy = \pi\cdot \text{area}\left(f\left(\mathbb{D}\cap\mathbb{D}_{1/2}(p)\right)\right)$$

WHICH IS FINITE, SINCE IT LIES IN A BOUNDED REGION OF $\mathbb{C}$.

SINCE $(L_r)^2/r$ IS INTEGRABLE, THERE IS A DECREASING SEQUENCE $r_n \to 0$ WITH $L_{r_n} \to 0$.

IF NOT, WE COULD FIND $c>0$ AND $0<\delta<1/2$ SO THAT $L_r > c$ FOR ALL $0<r<\delta$

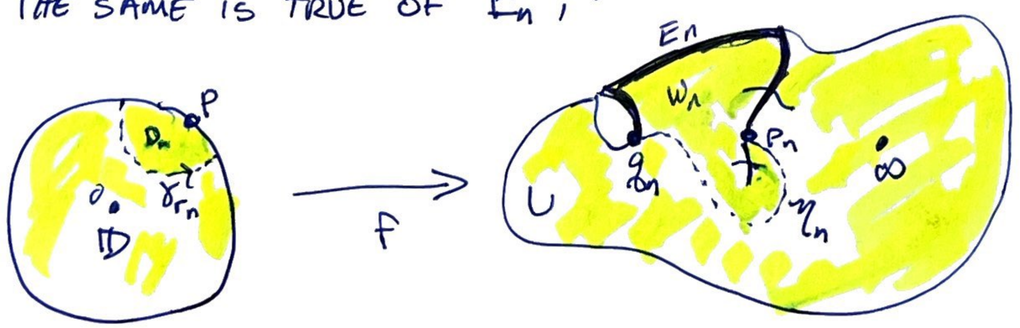
AND SO $\displaystyle\int_0^{1/2} \frac{(L_r)^2}{r}\,dr \geq \int_0^{\delta}\frac{(L_r)^2}{r}\,dr \geq c^2\int_0^{\delta}\frac{dr}{r} = +\infty$

SINCE L_{r_n} IS FINITE, $\eta_n = f\circ\gamma_{r_n}$ IS A CROSS-CUT IN U, WITH ~~LET~~ p_n, q_n ~~BE~~ THE LANDING POINTS IN ∂U.

OBSERVE $|p_n - q_n| \leq \text{diam}\{\eta_n\} \leq L_{r_n}$, SO BOTH $\to 0$ AS $n\to\infty$

SINCE U IS LOCALLY CONNECTED, FOR EACH n WE HAVE A CONNECTED $E_n \subset \partial U$ WITH $p_n, q_n \in E_n$ AND $\text{diam}(E_n) \to 0$.

THE SAME IS TRUE OF $\overline{E_n}$, SO LETS JUST LET E_n BE CLOSED.



NOW LET $D_n = \mathbb{D}_{r_n}(p) \cap \mathbb{D}$ AND W_n BE THE BOUNDED COMPONENT OF $U \setminus \{\eta_n\}$. SINCE $D_n \not\ni 0$, $f(D_n) \not\ni \infty$, SO $f(D_n) = W_n$

W_n IS CONTAINED IN A BOUNDED COMPONENT OF $\hat{\mathbb{C}} \setminus (\{\eta_n\} \cup E_n)$, OTHERWISE WE COULD JOIN $z \in W_n$ TO ∞ BY AN ARC AVOIDING $\{\eta_n\} \cup E_n$, AND THEN ANOTHER ARC FROM ∞ TO z CROSSING $\{\eta_n\} \cup E_n$ ONLY ONCE. THIS WOULD BE A JORDAN CURVE SEPARATING p_n FROM q_n, SO E_n WOULD NOT BE CONNECTED.

THUS $\operatorname{diam}(W_n) \leq \operatorname{diam}(\{\eta_n\} \cup E_n\} \to 0$ AS $n \to \infty$

SINCE $\overline{W_n} \supset \overline{W_{n+1}}$ ARE COMPACT, $\bigcap_{n=1}^{\infty} \overline{W_n}$ IS A SINGLE POINT $x \in \partial U$ WITH THE LIMIT x INDEPENDENT OF THE SEQ r_n.

THUS f EXTENDS TO $\bar{f} : \overline{\mathbb{D}} \to \overline{U}$.

TO SEE CONTINUITY OF $\bar{f}$, LET $p \in \partial \mathbb{D}$ AND $\varepsilon > 0$. LET $D_n = \mathbb{D}_{r_n}(p) \cap \mathbb{D}$ BE A NBHD OF p, AS BEFORE AND n LARGE ENOUGH THAT $\operatorname{diam}(f(D_n)) < \varepsilon$. IF $z \in D_n$, THEN $|f(z) - f(p)| < \varepsilon$.

IF $z \in \overline{D_n} \cap \partial \mathbb{D}$, CHOOSE ANOTHER NBHD OF z $D_m' = \mathbb{D}_{r_m'}(z) \cap \mathbb{D}$ WITH $D_m' \subset D_n$ SO $f(D_m') \subset W_n$ AND AGAIN $|f(z) - f(p)| < \varepsilon$.

FOR THE COROLLARY, WE NEED TWO LEMMAS.

LEMMA: LET γ, η BE JORDAN CURVES IN THE PLANE MEETING AT ONE POINT. THEN

$\operatorname{int}(\gamma) \cap \operatorname{int}(\eta) = \emptyset$ OR γ AND η ARE NESTED.

Pf/ SPOZE NOT. THEN $\mathrm{int}(\gamma)\cap\mathrm{ext}(\eta)$ IS NONEMPTY, AS IS $\mathrm{int}(\gamma)\cap\mathrm{int}(\eta)$ SO PART OF $\{\eta\}$ IS INTERIOR TO $\{\gamma\}$ AND PART EXTERIOR.
BUT THEN SINCE η AND γ MEET AT THE UNIQUE POINT ξ AND $\{\eta\}\setminus\{\xi\} = \{\eta\}\cap\mathrm{int}(\gamma) \amalg \{\eta\}\cap\mathrm{ext}(\gamma)$ IS DISCONNECTED.
BUT IT IS HOMEOMORPHIC TO AN INTERVAL ... $\Rightarrow\Leftarrow$

LEMMA: LET $I\subset\partial\mathbb{D}$ BE AN OPEN ARC AND $f\in\mathcal{O}(\mathbb{D})$ WHICH EXTENDS CONTINUOUSLY TO $\bar{f}$ ON $\overline{\mathbb{D}}$ WITH $f(I)=0$. THEN f IS IDENTICALLY 0 ON I.

Pf/ LET φ BE MÖBIUS WITH $\varphi(\mathbb{H})=\mathbb{D}$ AND $\varphi((-1,1))=I$

THEN $g=f\circ\varphi\in\mathcal{O}(\mathbb{H})$

AND EXTENDS CONTINUOUSLY TO $\mathbb{H}\cup(-1,1)$, SENDING I TO 0.

LET $U=\mathbb{C}-\big((-\infty,-1]\cup[1,+\infty)\big)$ AND $G:U\to\mathbb{C}$

BY
$$G(z)=\begin{cases} g(z) & z\in\mathbb{H}\\ 0 & z\in(-1,1)\\ \overline{g(\bar z)} & \bar z\in\mathbb{H}\end{cases}$$
(THIS IS A SPECIAL CASE OF SCHWARZ REFLECTION PRINCIPLE [SEE NEXT PAGE])

G IS CONTINUOUS ON U AND HOLOMORPHIC IN $U\setminus\mathbb{R}$, AND SINCE LINES ARE REMOVABLE, $G\in\mathcal{O}(U)$. BUT THEN SINCE $G(z_n)=0$ FOR A SET WITH $\{z_n\}$ ACCUMULATING IN U,

$G(z)=0$ ON U, SO $g\equiv 0$ AND $f\equiv 0$ ∎

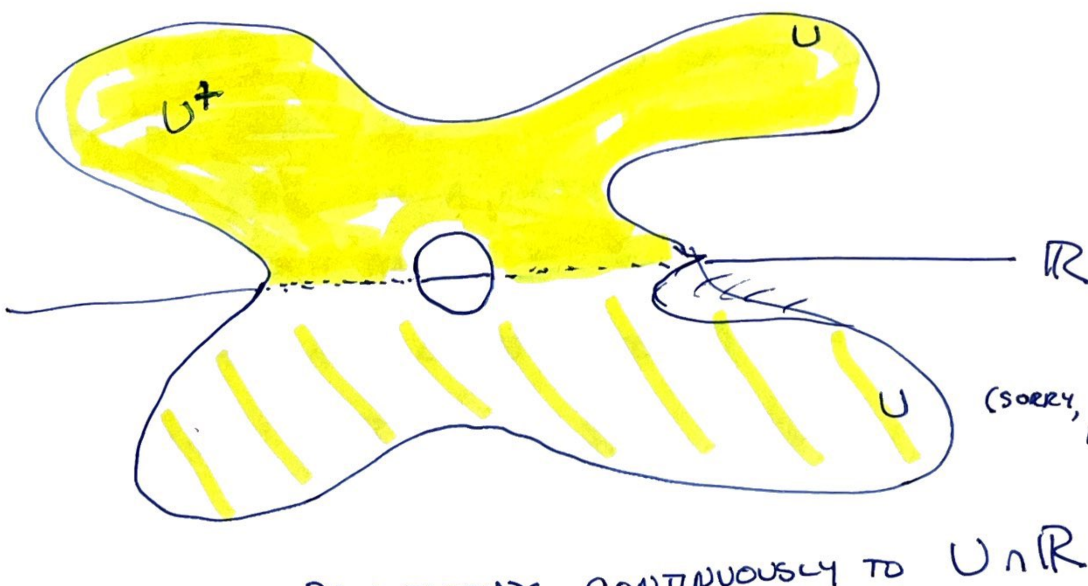
SCHWARZ REFLECTION PRINCIPLE

LET $U \subset \mathbb{C}$ BE REAL SYMMETRIC DOMAIN WITH $U^+ = \{z \in U \text{ with } \operatorname{Im}(z) > 0\}$ $I = U \cap \mathbb{R}$.

SPOZE $f \in \mathcal{O}(U^+)$ WITH $\operatorname{Im}(f(z)) \to 0$ AS $z \to I$ WITHIN U^+

THEN f EXTENDS UNIQUELY TO

$F \in \mathcal{O}(U)$ WHICH COMMUTES WITH $z \mapsto \bar{z}$



(SORRY, SHOULD BE SYMMETRIC

(THIS IS EASY IF $f(z)$ EXTENDS CONTINUOUSLY TO $U \cap \mathbb{R}$ FROM U^+, THE WEAKER ASSUMPTION THAT $\operatorname{Im}(f(z)) \to 0$ IS SUFFICIENT WHICH FOLLOWS FROM UNIQUENESS OF HARMONIC EXTENSIONS, WHICH I HOPE WE WILL GET TO NEXT WEEK)

Pf of Cor: $\Rightarrow$ / Let $f: \mathbb{D} \to U$ be a Riemann map. If f extends to a homeo $\bar{f}: \bar{\mathbb{D}} \to \bar{U}$, then $\partial U = f(\partial \mathbb{D})$ is certainly Jordan.

$\Leftarrow$ Now suppose ∂U is a Jordan curve. Then ∂U is locally connected as the continuous image of a locally connected set, so we just need to show $\bar{f}$ is injective, since by compactness of $\bar{\mathbb{D}}$, $\bar{f}^{-1}: \bar{U} \to \bar{\mathbb{D}}$ will be continuous.

So suppose there are points $P_1, P_2 \in \partial \mathbb{D}$ with $f(P_1) = f(P_2) = q \in \partial U$. We may assume $f(0) = 0$ and U is bounded in $\mathbb{C}$ (post-compose with Möbius sending some point in $\mathbb{C} \setminus \bar{U}$ to ∞ if not)

Take a crosscut in $\mathbb{D}$ avoiding 0 with ends at P_1 & P_2, and let D be the component of $\mathbb{D} \setminus \{\gamma\}$ not containing 0.

Then $f \circ \gamma$ is a crosscut in U avoiding 0, with both ends landing at q, and let $\{\eta\} = \{q\} \cup \{f \circ \gamma\}$.

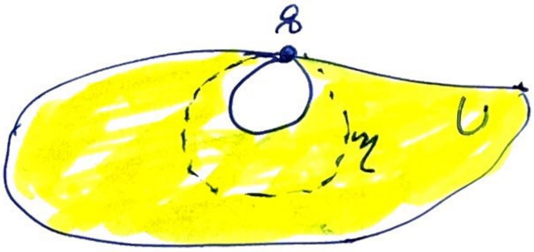
η is a Jordan curve whose interior intersects U, and since $\{\eta\} \cap U = q$, $\text{int}(\eta) \subset U$ since ∂U is also a Jordan curve.

Thus $f(D) = \text{int}(\eta)$, but this means $f(I) \subset \{\eta\} \cap \partial U$ so $f(I) = q$.

Thus $f - q$ which is holo in $\mathbb{D}$, is the constant 0, so $f(\mathbb{D}) = q$ $\Rightarrow \Leftarrow$.

NOTE

IN THE PROOF, IT IS ESSENTIAL THAT ∂U IS JORDAN SINCE η MAY PINCH U AT SOME POINT, AS IN THE FIGURE BELOW WHERE $\text{int}(\eta) \not\subset U$.



NOTE THAT IF ∂U IS LOCALLY CONNECTED, $\lim_{z \to p} f(z)$ WITH $|p|=1$ EXISTS. WHEN DOES THIS HOLD FOR ∂U NOT LOCALLY CONNECTED?

DEF: f A RIEMANN MAP HAS A RADIAL LIMIT AT $p \in \partial \mathbb{D}$ IF

$$f^*(p) = \lim_{r \to 1} f(rp)$$

EXISTS, THAT IS, THE IMAGE OF THE RADIAL LINE AT p LANDS AT A WELL DEFINED POINT IN $\hat{\mathbb{C}}$.

THM (FATOU, 1906) FOR EVERY RIEMANN MAP $f: \mathbb{D} \to U$, THE RADIAL LIMIT f^* EXISTS ALMOST EVERYWHERE ON $\partial \mathbb{D}$

EX:

Pf FOLLOWS FROM ANOTHER LENGTH-AREA ESTIMATE.

AGAIN, ASSUME $f(0)=\infty$, SO $f(\mathbb{D}-\mathbb{D}_{1/2})$ IS BOUNDED. FOR $t\in[0,2\pi]$, LET $L(t)$ BE THE LENGTH OF THE RADIAL CURVE, SO

$$(L(t))^2 = \left(\int_{1/2}^{1} |f'(re^{it})|\,dr\right)^2 \le \frac{1}{2}\int_{1/2}^{1} |f(re^{it})|^2\,dr \le \int_{1/2}^{1} |f'(re^{it})|^2\,r\,dr$$

THEN INTEGRATE OVER t:

$$\int_0^{2\pi} (L(t))^2\,dt \le \int_0^{2\pi}\int_{1/2}^{1} |f'(re^{it})|^2\,r\,dr\,dt$$

$$= \text{area}\left(f\left(\tfrac{1}{2}<|z|<1\right)\right) < +\infty.$$

SO $L(t)$ FINITE a.e.

SO $f^*(e^{it})$ EXISTS.