4/7 (1)

LAST TIME, INTRODUCED SCHLICHT FUNCTIONS

$$\mathcal{S} = \{ f : \mathbb{D} \to \mathbb{C} \mid f \in \mathcal{O}(\mathbb{D}),\ f \text{ injective},\ f(0)=0,\ f'(0)=1 \}$$

ALSO KNOWN AS UNIVALENT FUNCTIONS.

IE $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \qquad |z|<1$

$\boxed{\mathcal{S} \text{ IS COMPACT}}$

THE KOEBE FUNCTION $K : \mathbb{D} \to \mathbb{C} \setminus (-\infty, 1/4]$

IS $K(z) = \sum_1^\infty n z^n = \dfrac{z}{(1-z)^2} = \dfrac{1}{4}\left(\dfrac{1+z}{1-z}\right)^2 - 1/4$

RELATED ARE THE INVERTED SCHLICHT MAPS $\hat{\mathcal{S}}$:

$\hat{f}(z) = \dfrac{1}{f(1/z)}$, WITH LAURENT SERIES $z + \sum_{n=0}^{\infty} \dfrac{b_n}{z^n}, \quad |z|>1$

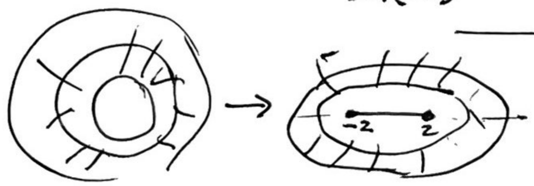
IF $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \qquad |z|<1$

THEN $\hat{f}(\omega) = \omega - a_2 + \dfrac{a_2^2 - a_3}{\omega} + \cdots \qquad |\omega|>1$

(BY A CALCULATION)

THE ZHUKOVSKII MAP

$$Z(\omega) = \omega + 1/\omega \in \hat{\mathcal{S}} = \hat{K}(\omega) + 2$$



MAPS $\mathbb{C} \setminus \overline{\mathbb{D}}$ TO $\mathbb{C} \setminus [-2,2]$

ALSO $\mathbb{D}$ TO

THM: GRÖNWALL AREA THEOREM

LET $\psi \in \hat{S}$ HAVE LAURENT SERIES $\psi(w) = w + \sum_{n=0}^{\infty} \frac{b_n}{w^n}$ $|w| > 1$ WITH $\psi(\hat{\mathbb{C}} - \overline{\mathbb{D}}) = U$.

THEN $\text{AREA}(\hat{\mathbb{C}} - U) = \pi\left(1 - \sum_{n=1}^{\infty} n|b_n|^2\right)$

SKETCH OF PROOF:

FOR $r > 1$, LET $\gamma_r(t) = \psi(re^{it})$.

BY GREEN'S THM $\frac{1}{2\pi i}\int_{\gamma_r} \bar{z}\,dz = \text{AREA}(\text{int}(\gamma_r)) = \text{AREA}(E_r)$

WRITE LAURENT SERIES FOR $\gamma_r(t)$ AND $\gamma_r'(t)$ WHICH CONVERGE UNIFORMLY ON $[0, 2\pi]$

AND INTEGRATE TERM BY TERM TO GET

$$\text{AREA}(E_r) = -\pi \sum_{n=-1}^{\infty} n|b_n|^2 r^{-2n}$$

THEN

$$\text{AREA}(\mathbb{C} - U) = \lim_{r \to 1} \text{AREA}(E_r) = \pi\left(1 - \sum_{n=1}^{\infty} n|b_n|^2\right).$$

COR:

- $|b_n| < \frac{1}{\sqrt{n}}$ FOR $n \geq 1$
- $|b_n| = 1 \iff \psi(w) = b_0 + \alpha Z\left(\frac{w}{\alpha}\right)$ WITH $|\alpha| = 1$

  $\iff \hat{\mathbb{C}} \setminus U$ IS A CLOSED LINE SEGMENT OF LENGTH 4 CENTERED AT b_0

PF OF COR

SINCE AREA$(\hat{\mathbb{C}} \setminus U) \geq 0$, GRÖNWALL'S SAYS

$\sum_{n=1}^{\infty} n|b_n|^2 < 1$, so $b_1 \leq 1$ (BUT ALSO $|b_n| < \frac{1}{\sqrt{n}}$)

IF $|b_1| = 1$, $b_n = 0$ FOR $n > 1$, SO $\psi(\omega) = \omega + b_0 + \frac{b_1}{\omega}$

THM: BIEBERBACH (1916) – DEBRANGES (1984)

IF $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ IS SCHLICHT THEN $|a_n| \leq n$

WITH EQUALITY FOR SOME $n \iff f$ IS ROTATIONALLY CONJUGATE TO THE KOEBE FUNCTION K

THE BIEBERBACH INEQUALITY (1916) ~~IS $|a_n| \leq 2$, IN THE~~ (IS JUST FOR a_2 WITH CONJECTURE FOR ALL n) IN 1984, deBranges PROVIDED THE PROOF FOR $n \geq 3$.

PROOF: AN IDEA WOULD BE TO APPLY THE COROLLARY ABOVE WITH $n = 1$ TO $\hat{f} \in \hat{S}$, WHICH HAS THE SERIES

$$\hat{f}(\omega) = \omega - a_2 + \frac{a_2^2 - a_3}{\omega} + \cdots$$

BUT THAT ONLY GIVES $|a_2^2 - a_3| \leq 1$

INSTEAD, WE MODIFY f A LITTLE AND TAKE A SQUARE ROOT. SPECIFICALLY, WRITE

$$f(z) = z f_1(z) = z\left(1 + a_2 z + a_3 z^2 + \cdots\right)$$

OBSERVE THAT SINCE $f(0) \neq 0$ IN $\mathbb{D}^*$ AND $f_1(0) = 1$, $f_1(z)$ HAS A HOLO. SQUARE ROOT $h \in \mathcal{O}(D)$ WITH $h(0) = 1$.

NOW LET $g(z) = z h(z^2)$. BUT

$$f(z^2) = z^2 f_1(z^2) = z^2 \left(h(z^2)\right)^2 = \left(g(z)\right)^2$$

<u>CLAIM</u> g IS SCHLICHT.

CERTAINLY $g \in \mathcal{O}(\mathbb{D})$ WITH $g(0) = 0$ $g'(0) = h(0) = 1$

g IS INJECTIVE:

SPOZE $g(z) = g(w)$ FOR SOME $z, w \in \mathbb{D}$.
THEN $f(z^2) = f(w^2)$, SO $z = \pm w$ SINCE f INJECTIVE

IF $z = w$, DONE.
IF $z = -w$, $g(z) = z h(z^2) = -w h(w^2) = -g(w)$
SO $g(z) = g(w) = 0$.
BUT SINCE $(g(z))^2 = f(z^2)$ AND f UNIVALENT $z = w = 0$.

SINCE $g(z) = z h(z^2)$, THE POWER SERIES HAS ONLY ODD POWERS.

$$g(z) = z + c_3 z^3 + \cdots$$

SO $(g(z))^2 = z^2 + 2c_3 z^4 + \cdots = f(z^2) = z^2 + a_2 z^4 + \cdots$

IE, $g(z) = z + \frac{a_2}{2} z^3 + \cdots$

SO $\hat{g}(w) = w - \frac{a_2}{2w} + \cdots$ SO $|a_2| \leq 2$

NOTE THAT $|a_2| = 2 \iff \hat{g}(\omega) = \lambda Z\left(\frac{\omega}{\lambda}\right)$ FOR $|\lambda| = 1$

$= \omega + \lambda^2/\omega$ (BY THE COR TO GRÖNWALL)

LET $\alpha = -1/\lambda^2$ SO THAT

$$\hat{g}(\omega) = \omega - 1/\alpha\omega, \Rightarrow g(z) = \frac{1}{\hat{g}(1/z)} = \frac{1}{1/z - z/\alpha} = \frac{z}{1 - z^2/\alpha}$$

NOW

$$f(z^2) = (g(z))^2 = \frac{z^2}{(1 - z^2/\alpha)^2}$$

SO

$$f(z) = \frac{z}{(1 - z/\alpha)^2} = \alpha K(\alpha^{-1} z)$$

, IE A ROTATION OF THE KOEBE MAP.

DE BRANGES PROOF THAT $|a_n| \le n$ ~~(WITH EQUALITY FOR~~ IS TOO MUCH TO DO HERE.

BIEBERBACH'S RESULT ~~THIS~~ GIVES US THE KOEBE 1/4 THM (CONJECTURED BY KOEBE IN 1907, PF BY BIEBERBACH IN 1916)

KOEBE 1/4 THM IF f IS SCHLICHT, THEN $\mathbb{D}_{1/4}(0) \subset f(\mathbb{D})$

Pf/ LET $p \notin f(\mathbb{D})$; WE SHOW $|p| \ge 1/4$.

$\varphi(z) = \frac{pz}{p - z}$ FIXES 0 AND SENDS $p \to \infty$, WITH $f'(0) = 1$

SO $\varphi \circ f \in \mathcal{S}$. BUT

$$\varphi \circ f(z) = \frac{p f(z)}{p - f(z)} = \frac{p(z + a_2 z^2 + \cdots)}{p - (z + a_2 z^2 + \cdots)} = \frac{z + a_2 z^2 + \cdots}{1 - \left(\frac{z}{p} + \frac{a_2}{p} z^2 + \cdots\right)}$$

$$= (z + a_2 z^2 + \cdots)\left(1 + \frac{z}{p} + \cdots\right) = z + (a_2 + 1/p) z^2 + \cdots$$

BUT $|a_2| \le 2$ AND $\left|a_2 + \frac{1}{p}\right| \le 2 \Rightarrow \frac{1}{|p|} = \left|a_2 + \frac{1}{p} - a_2\right| = \left|a_2 + \frac{1}{p}\right| + |a_2| \le 4$

IE $|p| \ge 1/4$

FROM THIS, MANY USEFUL DISTORTION BOUNDS FOLLOW.

THM: KOEBE DISTORTION LET f BE SCHLICHT. THEN IF $z \in \mathbb{D}$

- $$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}$$
- $$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3}$$

THE PROOF OF THE ABOVE LONG PREDATE DE BRANGES, BUT THE UPPER BOUNDS FOLLOW EASILY:

SINCE $f(z) = z + \sum_1 a_n z^n$ WITH $|a_n| < n$,

$$|f(z)| \leq |z| + \sum_1 |a_n||z|^n \leq \sum_1 n|z^n| = K(|z|) = \frac{|z|}{(1-|z|)^2}$$

AND

$$|f'(z)| \leq 1 + \sum_1 n|a_n||z|^{n-1} \leq \sum_1 n^2 |z|^{n-1} = K'(|z|) = \frac{1+|z|}{(1-|z|)^3}.$$

THE LOWER BOUNDS (OR NON-DEBRANGES PROOFS) TAKE ABOUT 2 PAGES AND I'M SKIPPING THEM HERE. NOTHING HARD, JUST....

ALSO SOMETIMES USEFUL IS

THM: IF $f \in \mathcal{S}$, WITH $|z| < 1$, THEN

$$\frac{1-|z|}{1+|z|} \leq \left| z\frac{f'(z)}{f(z)} \right| \leq \frac{1+|z|}{1-|z|}$$

SOME RELATED RESULTS WORTH MENTIONING.

CONSIDER MAPS $f: \mathbb{D} \to \mathbb{C}$ WITH $f \in \mathcal{O}(\mathbb{D})$

$f'(0) = 1$ BUT $f \notin \mathcal{S}$ NECESSARILY.

LANDAU'S THM: THERE IS A CONSTANT $L > 0$ SUCH THAT IF $f \in \mathcal{O}(\mathbb{D})$ WITH $f'(0) = 1$, $f(\mathbb{D})$ CONTAINS A DISK OF RADIUS L

LANDAU'S CONST.

THE EXACT VALUE OF L IS STILL UNKNOWN BUT

$$\frac{1}{2} < L \leq L_0 = \frac{\Gamma(1/3)\,\Gamma(\frac{5}{6})}{\Gamma(\frac{1}{6})} \approx 0.544\ldots \quad \left(\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,dt\right)$$

(1995 & YANAGIHARA): $B > \frac{1}{2} + 10^{-335}$)

BLOCH'S THM THERE IS A $B > 0$ SO THAT ~~WITH~~ $f'(0) = 1$, $f \in \mathcal{O}(\mathbb{D})$, ~~THEN $f(\mathbb{D})$ CONTAINS~~ THEN THERE IS A REGION $V \subset \mathbb{D}$ WITH f INJECTIVE ON V WITH $f(V)$ CONTAINING A DISK OF RADIUS B

BLOCH CONSTANT

CURRENTLY KNOWN:

$$0.433 < \frac{\sqrt{3}}{4} < B \leq B_0 = \frac{1}{\sqrt{1+\sqrt{3}}} \cdot \frac{\Gamma(\frac{1}{3})\,\Gamma(\frac{11}{12})}{\Gamma(\frac{1}{4})} \approx 0.4719\ldots$$

GAUTHIER & CHEN $\longrightarrow + 2 \cdot 10^{-4}$

FINCH (2003)

AHLFORS – GRUNSKY CONJECTURE (1936)

NOW WE CONSIDER WHAT HAPPENS TO $f(z)$ AS $|z| \to 1$ FOR A RIEMANN MAP f.

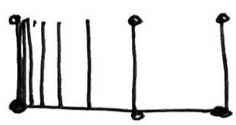
THM: (CARATHÉODORY 1913). A RIEMANN MAP $f: \mathbb{D} \to U$ EXTENDS TO A CONTINUOUS MAP FROM $\overline{\mathbb{D}}$ TO $\overline{U}$ $\iff$ ∂U IS LOCALLY CONNECTED

RECALL THAT $X \subset \hat{\mathbb{C}}$ IS LOCALLY CONNECTED $\iff$ EACH POINT $p \in X$ HAS ARBITRARILY SMALL CONNECTED NEIGHBORHOODS.

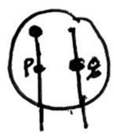
IE FOR EACH $p \in X$ AND EVERY OPEN NBHD Ω OF p, THERE IS AN OPEN SUBSET $V \subset \Omega$ WITH $p \in V$ AND $V \cap X$ CONNECTED.

STANDARD EXAMPLE: THE COMB SPACE $X \subset \mathbb{C}$

$$X = [0,1] \cup \{ z \in \mathbb{C} \mid 0 \le \mathrm{Im}(z) \le 1 \text{ AND } \mathrm{Re}(z) \in \{0, 1/n\} \text{ WITH } n \in \mathbb{N} \}$$



IS NOT LOCALLY CONNECTED FOR ANY $p \in X$ WITH $\mathrm{Im}(p) > 0$ AND $\mathrm{Re}(p) = 0$.

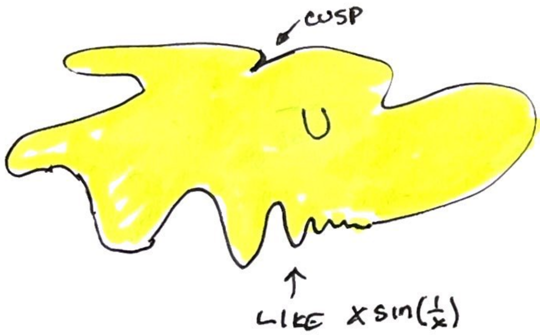


SINCE IF $q \in X \cap \Omega$ WITH $\mathrm{Re}(q) > 0$ THE SEGMENT CONTAINING p IS $\left\{ \begin{matrix} \text{DISJOINT} \\ \text{IN } \Omega \end{matrix} \right\}$ FROM THE ONE CONTAINING q.

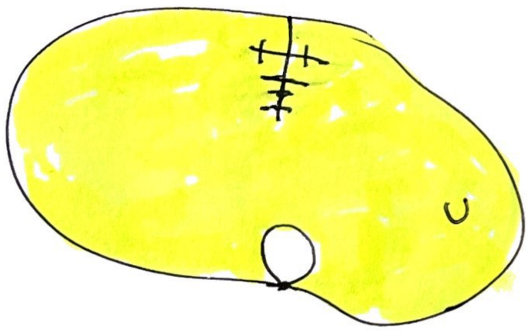
A SHARPER VERSION:

THM: A RIEMANN MAP $\mathbb{D} \to U$ EXENDS TO A HOMEOMORPHISM $\overline{\mathbb{D}} \to \overline{U}$ IF AND ONLY IF ∂U IS A JORDAN CURVE

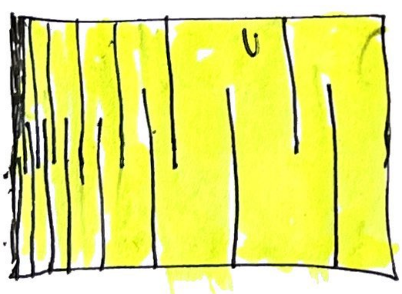
Some Examples



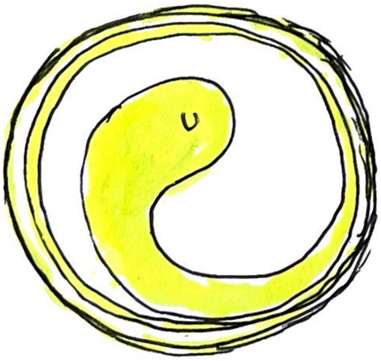
f EXTENDS TO HOMEO
$\overline{\mathbb{D}} \to \overline{U}$



f EXTENDS CONTINUOUSLY
$\overline{\mathbb{D}} \to \overline{U}$ BUT NOT HOMEO
(SINCE POINTS WITH 2 OR 4 PREIMAGES)



LOCAL CONNECTIVITY FAILS ON LEFT SIDE, SO f EXTENDS CONTINUOUSLY AT ALL POINTS <u>EXCEPT</u> LEFT EDGE OF $\overline{U}$.



LOCAL CONNECTIVITY FAILS AT EACH POINT OF THE LIMITING CIRCLE, SO f CANNOT EXTEND CONTINUOUSLY THERE.