TERMINOLOGY: THE PHRASES "BIHOLOMORPHISM" AND "CONFORMAL MAP" ARE INTERCHANGABLE (ALSO "CONFORMAL ISOMORPHISM").

THE RIEMANN MAPPING THM

EVERY SIMPLY CONNECTED DOMAIN $U \subsetneq \mathbb{C}$ IS CONFORMALLY ISOMORPHIC TO $\mathbb{D}$

MORE SPECIFICALLY, IF $U \subsetneq \mathbb{C}$ IS SIMPLY CONNECTED, OPEN WITH $p \in U$, THERE IS A CONFORMAL MAP $f: U \to \mathbb{D}$ WITH $f(p) = 0$ AND $f(U) = \mathbb{D}$.

f IS UNIQUE IF WE ALSO REQUIRE $f'(0) > 0$ (IE, UP TO RIGID ROTATION OF $\mathbb{D}$)

CONJECTURED & PARTIALLY PROVEN BY RIEMANN (1851). LATER PROVEN BY ~~OSGOOD~~ SCHWARZ (1870), OSGOOD (1900) KOEBE (1907) CARATHÉODORY (1912). MODERN PROOF BY RIESZ & FEJÉR (1923).

EQUIVALENT STATEMENT:

~~IF THERE ARE TWO DISTINCT POINTS $p, q \in \hat{\mathbb{C}}$~~

IF U IS A SIMPLY CONNECTED DOMAIN IN $\hat{\mathbb{C}} \setminus \{p, q\}$ WITH $p \neq q$, THEN $U \cong \mathbb{D}$ CONFORMALLY.

Pf/ LETS SETTLE THE UNIQUENESS FIRST. ②

SPOZE $f(U) = \mathbb{D} = g(U)$ WITH $f(p) = 0 = g(p)$, f, g CONFORMAL.
THEN $f \circ g^{-1}$ IS AN AUTOMORPHISM OF $\mathbb{D}$ FIXING 0,
SO IT IS A RIGID ROTATION $z \mapsto \alpha z$ BY THE SCHWARZ LEMMA. THUS $g = \alpha f$, $|\alpha| = 1$ (AND, IN PARTICULAR,
WE CAN REQUIRE $f'(p)$ TO BE POSITIVE.

THE MAIN PART OF THE PROOF IS TO CONSIDER THE FAMILY $\mathcal{F} = \{ f : U \to \mathbb{D} \mid f(p) = 0,\ f \text{ INJECTIVE \& HOLOMORPHIC} \}$.

- FIRST, WE SHOW $\mathcal{F}$ NONEMPTY.

SINCE $U \neq \mathbb{C}$, THERE IS SOME $a \in \mathbb{C}$ WITH $a \notin U$.

SINCE U IS SIMPLY CONNECTED, $z \mapsto z - a$ HAS A HOLOMORPHIC SQUARE ROOT, h IN U, THAT IS,

$$h \in \mathcal{O}(U) \text{ WITH } h^2(z) = z - a.$$

FURTHER $h(U) \cap -h(U) = \emptyset$.

IF $h(z) = -h(w)$ FOR SOME $z, w \in U$,
THEN $z - a = h^2(z) = h^2(w) = w - a$, SO $z = w$.

TAKE $g \in -h(U)$ AND A DISK $D = \mathbb{D}_r(g) \subset -h(U)$.
THEN $\varphi : z \mapsto \dfrac{r}{z - g}$ HAS $\varphi(D) = \mathbb{C} - \overline{D}$,
SO $\varphi \circ h(U) \subset \mathbb{D}$. LET $\psi \in \mathrm{Aut}(\mathbb{D})$ SEND $\psi(\varphi(h(p))) = 0$

So $\psi \circ \varphi \circ h \in \mathcal{F}$.

- THE KOEBE/CARATHEODORY SQUARE ROOT TRICK

IF $f \in \mathcal{F}$ WITH $f(U) \neq \mathbb{D}$, WE FIND $g \in \mathcal{F}$ WITH $|g'(p)| > |f'(p)|$.

LET $a \in \mathbb{D} \setminus f(U)$. $\varphi_a(z) = \dfrac{z-a}{1-\bar{a}z} \in \mathrm{Aut}(\mathbb{D})$ SENDS $a \to 0$, WITH INVERSE $\varphi_{-a}(z)$.

SO $(\varphi_a \circ f) : U \to \mathbb{D}$ IS INJECTIVE AND OMITS 0, SO IT HAS A HOLOMORPHIC SQUARE ROOT h IN U,

IE $s \circ h = \varphi_a \circ f$ WITH $s(z) = z^2$.

h IS INJECTIVE, $h(U) \subset \mathbb{D}$.

LET $b = h(p)$, SO $g = \varphi_b \circ h \in \mathcal{F}$

BUT

$$f = \varphi_{-a} \circ s \circ h = \varphi_{-a} \circ s \circ (\varphi_{-b} \circ g)$$

OBSERVE THAT $\varphi_{-a} \circ s \circ \varphi_{-b} = \psi$ SATISFIES $\psi(\mathbb{D}) = \mathbb{D}$, $\psi(0) = 0$

AND ψ IS <u>NOT</u> INJECTIVE, SO BY THE SCHWARZ LEMMA $|\psi'(0)| < 1$.

BUT $|f'(p)| = |\psi'(0)|\,|g'(p)| < |g'(p)|$,

AS WE WANTED ☺

- SOLVE THE EXTREMAL PROBLEM IN $\mathcal{F}$ (4)

WE WANT TO FIND THE $f \in \mathcal{F}$ THAT MAXIMIZES $|f'(p)|$
(AND SHOW IT EXISTS...)

LET $M = \sup_{f_n \in \mathcal{F}} |f_n'(p)|$, AND OBSERVE $0 < M < +\infty$, SINCE

- EACH $f_n \in \mathcal{F}$ IS INJECTIVE, SO $|f_n'(p)| > 0$
- IF $\mathbb{D}_r(p) \subset U$, $f_n(\mathbb{D}_r(p)) \subset \mathbb{D}$, SO $|f_n'(p)| < \frac{1}{r}$

CONSIDER $f_n \in \mathcal{F}$ WITH $|f_n'(p)| \to M$.

BY MONTEL'S THEOREM, SINCE $\mathcal{F}$ IS UNIFORMLY BOUNDED (EACH ELEMENT HAS ITS IMAGE IN $\mathbb{D}$), THERE IS A SUBSEQUENCE $f_{n_j} \to f$ COMPACTLY IN U, WITH $f \in \mathcal{O}(U)$.

WE NEED TO SHOW $f \in \mathcal{F}$ AND $f(U) = \mathbb{D}$.

CERTAINLY $f(p) = 0$ AND $f(U) \subset \overline{\mathbb{D}}$.
BY THE OPEN MAPPING THEOREM, IF f TAKES A VALUE ON $\partial\mathbb{D}$, THEN f IS CONSTANT. BUT $|f'(p)| = M > 0$, SO f IS NOT CONSTANT

BUT SINCE EACH f_n IS INJECTIVE IN U AND f IS NONCONSTANT, WE KNOW BY THE COROLLARY TO HURWITZ THAT f IS INJECTIVE. SO $f \in \mathcal{F}$.

$f(U) = \mathbb{D}$ SINCE IF f OMITTED A VALUE OF $\mathbb{D}$, WE COULD APPLY THE PREVIOUS STEP AGAIN TO GET A NEW FUNCTION WITH LARGER DERIVATIVE AT p

GIVEN ANY SIMPLY CONNECTED $U \subsetneq \mathbb{C}$ (OR $\hat{\mathbb{C}} \setminus \{a,b\}$)
ANY CONFORMAL $f: \mathbb{D} \to U$ IS CALLED A RIEMANN MAP FOR U (BUT ALSO SOMETIMES ITS INVERSE, TOO)
AND GIVEN $p \in U$, THE RIEMANN MAP

$$f_p : \mathbb{D} \to U \quad \text{WITH } f_p(0) = p \quad f_p'(0) > 0$$

IS THE RIEMANN MAP OF U WITH CENTER p
(OR THE NORMALIZED RIEMANN MAP, OR MAYBE ITS INVERSE, ...)

~~A SPE~~ AN IMPORTANT CLASS OF RIEMANN MAPS ARE THE SCHLICHT FUNCTIONS:

DEF IF $f \in \mathcal{O}(\mathbb{D})$ IS INJECTIVE ~~AND~~ WITH $f(0) = 0, f'(0) = 1$
f IS A SCHLICHT FUNCTION (OR JUST "f IS SCHLICHT"

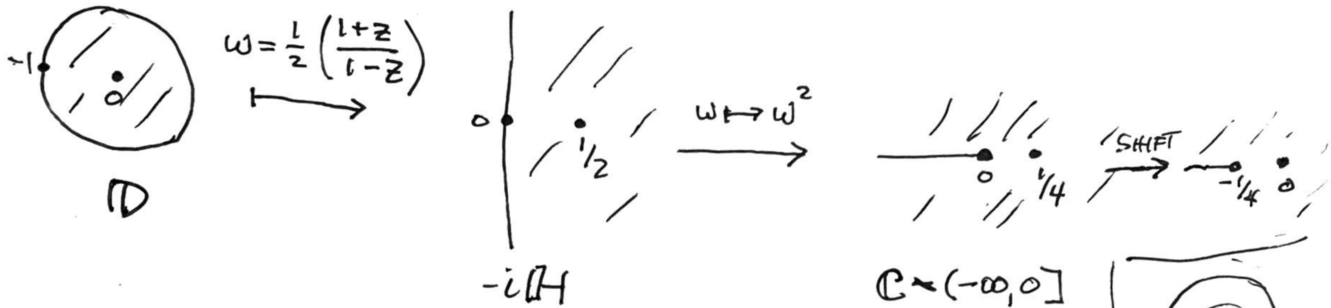
OFTEN DENOTED BY $f \in \mathcal{S}$ (SCHLICHT IS GERMAN FOR "PLAIN" OR "SIMPLE" OR "ORDINARY"

NOTE THAT $\mathcal{S}$ IS COMPACT: IF $f_n \in \mathcal{S}$ CONVERGES TO f,
$f \in \mathcal{S}$.

EXAMPLE: THE KOEBE FUNCTION $K(z) = \sum_{n=1}^{\infty} n z^n = \dfrac{z}{(1-z)^2}$
IS SCHLICHT.

$$K(\mathbb{D}) = \mathbb{C} \setminus (-\infty, -1/4]$$

TO SEE THIS,



SO

$$K(z) = \left(\frac{1}{2}\left(\frac{1+z}{1-z}\right)\right)^2 - \frac{1}{4} = \frac{z}{(1-z)^2}$$

SOMETIMES IT IS HELPFUL TO ~~look~~ EXAMINE A SCHLICHT FUNCTION f VIA THE INVERTED MAP

$\hat{f} : w \mapsto \dfrac{1}{f(1/w)}$, SINCE $\hat{f} : \hat{\mathbb{C}} - \overline{\mathbb{D}} \to \hat{\mathbb{C}} - K$

COMPACT IN $\mathbb{C}$ SO FINITE AREA.

DEF: $\hat{\mathcal{S}} = \{\hat{f} \mid f \in \mathcal{S}\}$.

$\psi \in \hat{\mathcal{S}}$ IS HOLOMORPHIC AND INJECTIVE IN $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ WITH LAURENT SERIES $\psi(w) = w + \sum_{n=0}^{\infty} b_n w^{-n}$, $|w| > 1$

EX: THE ZHUKOVSKII MAP $Z : w \to w + 1/w$

$$Z(w) = 2 + \hat{K}(w) = w + 1/w$$

$Z : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus [-2,2]$ ALSO $Z : \mathbb{D} \to \hat{\mathbb{C}} \setminus [-2,2]$