Normal Families

Now we change our metric space from the Euclidean metric on $\mathbb{C}$ to the spherical metric on $\hat{\mathbb{C}}$. This means we no longer need to avoid ∞ as a limit. For example, in $\mathbb{C}$, the family of translations $\{z \mapsto z+\beta\}$ is not precompact, but in $\hat{\mathbb{C}}$, every sequence has a convergent subsequence ~~every subsequence~~ that either converges to a translation, to the identity, or to the constant function ∞.

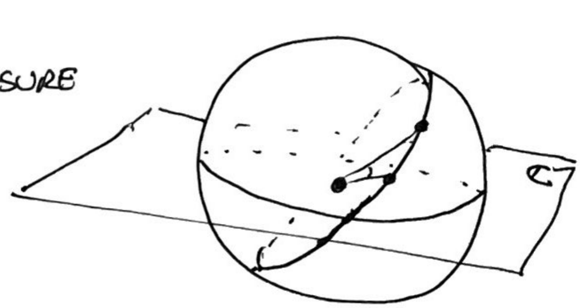
Recall that the spherical metric on $\hat{\mathbb{C}}$ is

$$\sigma = \frac{2}{1+|z|^2}\,dz$$

which is the standard metric on $S^2 \subset \mathbb{R}^3$ induced by stereographic projection.

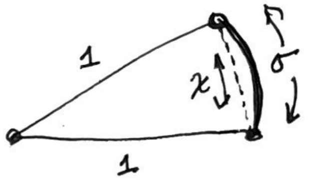
Minimal geodesics are arcs of great circles (ie, intersection of S^2 with a plane in $\mathbb{R}^3$ containing the center of S^2)

The distance $\mathrm{dist}_\sigma(z,w)$ is the measure of the angle formed by the two points of S^2 with vertex at the center.



A related metric is the <u>chordal metric</u>

$$\chi(z,w) = 2\sin\left(\tfrac{1}{2}\,\mathrm{dist}_\sigma(z,w)\right) = \begin{cases} \dfrac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}} & z,w\in\mathbb{C} \\ \dfrac{2}{\sqrt{1+|z|^2}} & w=\infty \end{cases}$$



IT IS ~~NOT~~ TEDIOUS BUT NOT HARD TO SHOW THAT

$$\frac{2}{\pi}\,\mathrm{dist}_\sigma(z,w) \le \chi(z,w) \le \mathrm{dist}_\sigma(z,w)$$

THAT IS, THE CHORDAL DISTANCE AND THE SPHERICAL DISTANCE ON $\hat{\mathbb{C}}$ ARE COMPARABLE

FURTHER, IN ANY BOUNDED SUBSET OF $\mathbb{C}$, THE EUCLIDEAN AND SPHERICAL DISTANCES ARE COMPARABLE

MORE EXPLICITLY,

$$\boxed{\frac{2}{1+R^2}|z-w| \le \mathrm{dist}_\sigma(z,w) \le 2|z-w| \qquad \text{WHERE } |z|,|w| \le R}$$

~~THIS FOLLOWS FROM THE FACT THAT INVERSION $z \mapsto 1/z$ IS AN ISOMETRY OF $\hat{\mathbb{C}}$~~

THESE ARE BOTH STRAIGHTFORWARD CALCULATIONS:

$$\mathrm{dist}_\sigma(z,w) \le \mathrm{length}_\sigma[z,w] = \int_{[z,w]} \frac{2|d\zeta|}{1+|\zeta|^2} \le \int_{[z,w]} 2|d\zeta| = 2|z-w|$$

AND

$$\mathrm{dist}_\sigma(z,w) \ge \chi(z,w) = \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}} \ge \frac{2|z-w|}{1+R^2}$$

FOR ANY MEROMORPHIC FUNCTION ON $U \subset \mathbb{C}$,
WE CAN THINK OF THIS AS A HOLOMORPHIC MAP $U \to \hat{\mathbb{C}}$.

SO WE CAN CONSIDER $\mathcal{M}(U)$ AS A SUBSPACE OF $C(U, \hat{\mathbb{C}})$
WITH THE COMPACT CONVERGENCE TOPOLOGY (WHERE $\hat{\mathbb{C}}$ HAS THE SPHERICAL METRIC)

THE ANALOGUE OF THE WEIERSTRASS THM FROM LAST WEEK IS

$$\boxed{\text{THM: LET } U \subset \mathbb{C} \text{ BE A DOMAIN, AND } f_n \in \mathcal{M}(U) \text{ WITH } f_n \to f \text{ IN } \mathcal{C}(U, \hat{\mathbb{C}}). \text{ THEN } f \in \mathcal{M}(U) \text{ OR } f = \infty \text{ EVERYWHERE}}$$

Pf/ SPOZE f IS NOT IDENTICALLY ∞, AND $p \in U$.
~~FIRST~~, WE WANT TO SHOW f MEROMORPHIC ON A NBHD OF p.

FIRST, SUPPOSE $f(p) \neq \infty$. SO THEN THERE IS SOME DISK
$D = \mathbb{D}_r(p)$ WITH $\overline{\mathbb{D}_r} \subset U$ AND $\varepsilon > 0$ WITH $dist_\sigma(f(z), \infty) \geq 2\varepsilon$ FOR $z \in D$.

SINCE $f_n \to f$, FOR n SUFF. LARGE $dist_\sigma(f_n(z), f(z)) < \varepsilon$ FOR $z \in D$
AND THEN BY THE TRIANGLE INEQUALITY, $dist_\sigma(f_n(z), \infty) \geq \varepsilon$ ON D
THAT IS f_n HAS NO POLES ON D FOR n LARGE, IE

$$f_n \in \mathcal{O}(D).$$

SINCE BOTH f AND f_n MAP D INTO A BOUNDED SET (IE THE COMPLEMENT OF AN ε-NBHD OF ∞),
THE SPHERICAL DISTANCE DOMINATES THE EUCLIDEAN DISTANCE,
SO $f_n \to f$ IN $\mathcal{C}(U, \hat{\mathbb{C}}) \Rightarrow f_n \to f$ IN $\mathcal{C}(U, \mathbb{C})$ UNIFORMLY.

THUS $f \in \mathcal{O}(\mathbb{D})$.

MORE...

IF $f(p) = \infty$, SINCE INVERSION IS AN ISOMETRY IN $\hat{\mathbb{C}}$,

WE HAVE $\frac{1}{f_n} \to \frac{1}{f}$ IN $\mathcal{C}(U, \hat{\mathbb{C}})$ AND CAN USE

THE ABOVE ARGUMENT TO CONCLUDE THAT FOR A DISK $D = \mathbb{D}_r(p)$,

$\frac{1}{f_n} \in \mathcal{O}(D)$ FOR n LARGE AND $\frac{1}{f_n} \to \frac{1}{f}$ UNIFORMLY

IN THE EUCLIDEAN METRIC, SO $\frac{1}{f} \in \mathcal{O}(D)$.

SINCE f IS NOT IDENTICALLY INFINITY, $\frac{1}{f}$ IS NOT IDENTICALLY 0,

AND $f \in \mathcal{M}(D)$ WITH A POLE AT p. ∎

AN IMMEDIATE CONSEQUENCE OF THE PROOF IS

COR: SPOZE $f_n \to f$ IN $\mathcal{M}(U)$. THEN FOR EVERY COMPACT $K \subset U \setminus f^{-1}(\infty)$, FOR n LARGE ENOUGH f_n HAS NO POLES, AND $f_n \to f$, $f_n' \to f'$ UNIFORMLY ON K IN THE EUCLIDEAN METRIC

IN OTHER WORDS, AWAY FROM POLES OF f, EITHER THE POLES OF f_n ACCUMULATE ON THOSE OF f, OR THEY "DISAPPEAR" BY ACCUMULATING ON ∂U OR MARCHING OFF TO ∞.

EXAMPLE: LET $f_n = \frac{n^2}{(n-z)(nz-1)}$. AS $n \to \infty$, THESE LIMIT ON $\frac{1}{z}$

DIRECT COMPUTATION IN THE CHORDAL METRIC SHOWS THAT THE POLE OF f_n AT $\frac{1}{n} \to 0$ AND THE POLE AT $n \to \infty$

IN THE PREVIOUS THM, WE CAN SPECIALIZE TO HOLOMORPHIC FUNCTIONS

COR LET $U \subset \mathbb{C}$ BE A DOMAIN, $f_n \in \mathcal{O}(U)$ WITH $f_n \to f$ IN $\mathcal{C}(U, \hat{\mathbb{C}})$

THEN $f \in \mathcal{O}(U)$ OR f IS IDENTICALLY ∞ ON U.

IF $f \not\equiv \infty$, THEN $f_n \to f$ COMPACTLY IN U IN THE EUCLIDEAN METRIC

Pf/ SPOZE $f \not\equiv \infty$, SO $f \in \mathcal{M}(U)$ BY THE THM.

IF f HAS A POLE p, THEN AS IN THE Pf, WE CAN FIND $D = \mathbb{D}_r(p)$ SO THAT $1/f_n \in \mathcal{O}(D)$ AND $1/f_n \to 1/f$ UNIFORMLY IN THE EUCLIDEAN METRIC.

NOTE $1/f$ IS NOT IDENTICALLY 0 ~~SINCE~~ AND $1/f_n$ IS NONZERO IN D

SO BY HURWITZ'S THM, $1/f$ HAS NO ZEROS IN D. THIS CONTRADICTS $\frac{1}{f(p)} = 0$, SO f HAS NO POLES.

THUS $f \in \mathcal{O}(U)$ AND $f_n \to f$ COMPACTLY IN THE EUCLIDEAN METRIC.

AS MENTIONED BEFORE,

DEF A FAMILY $\mathcal{F} \subset \mathcal{M}(U)$ IS <u>NORMAL</u> IF IT IS RECOMPACT IN $\mathcal{C}(U, \hat{\mathbb{C}})$.

THAT IS A FAMILY OF MEROMORPHIC FUNCTIONS IS NORMAL IF EVERY SEQUENCE HAS A SUBSEQUENCE CONVERGING COMPACTLY IN U USING THE SPHERICAL METRIC IN THE IMAGE

EX $\{z^n\}$ $n \in \mathbb{Z}^+$ IS A NORMAL FAMILY IN $\mathbb{D}$ AND IN $\mathbb{C} \setminus \overline{\mathbb{D}}$

- IN $\mathbb{D}$ $z^n \to 0$ COMPACTLY
  IN $\mathbb{C} \setminus \overline{\mathbb{D}}$, $z^n \to \infty$. HOWEVER, THE FAMILY CANNOT BE NORMAL IN ANY NEIGHBORHOOD OF THE UNIT CIRCLE, SINCE ANY LIMIT f MUST BE DISCONTINUOUS.

THE <u>SPHERICAL DERIVATIVE NORM</u> WILL BE USEFUL IN WHAT FOLLOWS. THINKING OF $f(U,g) \to (\hat{\mathbb{C}}, \sigma)$ WITH g THE EUCLIDEAN NORM $|dz|$

$$f^{\#}(z) = \frac{1}{2} \|f'(z)\|_{g,\sigma} = \frac{|f'(z)|}{1+|f(z)|^2}$$

WE CAN EXTEND THIS TO POLES OF f BY CONTINUITY:
IF p IS A POLE OF ORDER m,
$$f^{\#}(p) = \begin{cases} \dfrac{1}{\lim_{z\to p}(z-p)f(z)} & m=1 \\ 0 & m>1 \end{cases}$$
SO $f^{\#}$ IS WELL DEFINED AND NON NEGATIVE, EVEN AT A POLE p.

- NOTE THAT $f^{\#}$ SATISFIES THE CHAIN RULE $(f \circ g)^{\#} = (f^{\#}\circ g)\,|g'(z)|$ IF g IS HOLOMORPHIC
- ALSO $\left(\frac{1}{f}\right)^{\#} = f^{\#}$ WHICH FOLLOWS FROM THE FACT THAT $\imath : z \to 1/z$ IS AN ISOMETRY IN $\hat{\mathbb{C}}$;

$$\left\|\left(\tfrac{1}{f}\right)'\right\|_{g,\sigma} = \|(\imath\circ f)'\|_{g,\sigma} = \|\imath' \circ f\|_{\sigma,\sigma} \cdot \|f'\|_{g,\sigma} = \|f'\|_{g,\sigma}$$

IN PARTICULAR, AT A POLE p, LET $g = \frac{1}{f}$ AND $f^{\#}(p) = g^{\#}(p) = \dfrac{|g'(p)|}{1+|g(p)|^2}$

MARTY'S THM (1931)

LET $U \subset \mathbb{C}$ BE A DOMAIN. THEN

$\mathcal{F} \subset \mathcal{M}(U)$ IS NORMAL $\Longleftrightarrow$ THE FAMILY $\mathcal{F}^{\#} = \left\{ f^{\#} = \frac{|f'|}{1+|f|^2}\ f \in \mathcal{F} \right\}$ OF SPHERICAL DERIVATIVE NORMS IS COMPACTLY BOUNDED.

Pf/ SPOZE $\mathcal{F}$ IS NORMAL BUT $\mathcal{F}^{\#}$ IS NOT COMPACTLY BOUNDED.

THEN THERE IS A COMPACT $K \subset U$ AND $\{z_n\} \subset K$ WITH $\{f_n\} \in \mathcal{F}$

SO THAT $\lim\limits_{n \to \infty} f_n^{\#}(z_n) = +\infty$.

SINCE K COMPACT, WE MAY PASS TO A SUBSEQ AND ASSUME $z_n \to p \in K$,

AND $f_n \to f$ IN $\mathcal{C}(U, \hat{\mathbb{C}})$.

BUT THEN BY THE THM, $f \in \mathcal{M}(U)$ OR $f = \infty$ EVERYWHERE.

IF f NOT IDENTICALLY ∞, WITH $f(p) \neq \infty$, WE TAKE $D = \mathbb{D}_r(p)$ SO THAT

$\bar{D} \subset U$ AND f HAS NO POLES IN D.

BUT FROM THE COR., f_n HAS NO POLES IN D FOR n LARGE

SO $f_n \to f$, $f_n' \to f'$ UNIFORMLY ON K IN THE EUCLIDEAN METRIC, SO

$$f_n^{\#}(z_n) = \frac{|f_n'(z_n)|}{1+|f_n(z_n)|} \longrightarrow \frac{|f'(p)|}{1+|f(p)|^2} = f^{\#}(p)$$

A CONTRADICTION.

IF $f(p) = \infty$, APPLY THE SAME ARGUMENT TO $g_n = \frac{1}{f_n}$ AND

CONCLUDE THAT $g_n^{\#}(p) \to g^{\#}(p)$, SO $f_n^{\#}(p) \to f^{\#}(p)$ AGAIN A CONTRA.

IF f IDENTICALLY ∞, $g_n = \frac{1}{f_n} \to 0$ IN $\mathcal{C}(U, \hat{\mathbb{C}})$, SO

$g_n \to 0$, $g_n' \to 0$ UNIFORMLY IN K IN THE EUCLIDEAN METRIC,

SO $f_n^{\#}(z_n) = g_n^{\#}(z_n) \to 0$ AGAIN CONTRA.

CONVERSELY, SPOZE $\mathcal{F}^{\#}$ IS COMPACTLY BOUNDED.

WE NEED TO SHOW $\mathcal{F}$ IS NORMAL, IE PRECOMPACT IN $\mathcal{C}(U, \hat{\mathbb{C}})$.

USING THE ARZELA-ASCOLI THM, IT IS SUFFICIENT TO SHOW EQUICONTINUITY.

TAKE $p \in U$ AND FIND $D = \mathbb{D}_r(p)$ WITH $\overline{D} \subset U$.

THERE IS $M > 0$ SO THAT $\|f'(\zeta)\|_{g,\sigma} = 2f^{\#}(\zeta) \leq M$ FOR ALL $\zeta \in D$ AND $f \in \mathcal{F}$ (SINCE $f^{\#}$ COMPACTLY BOUNDED)

BUT THEN $\text{dist}_\sigma(f(z), f(p)) \leq \sup_{z \in \mathbb{D}_r(p)} \|f'(p)\|_{g,\sigma} \cdot |z-p| \leq M|z-p|$

FOR ALL $z \in \mathbb{D}_r(p)$ AND $f \in \mathcal{F}$,

THAT IS f IS EQUICONTINUOUS ∎

EX: RETURNING TO THE EXAMPLE OF $\{z^n\}$,

$$f_n^{\#}(z) = \frac{n|z|^{n-1}}{1+|z^{2n}|}$$

IF $|z| \leq r < 1$, $f_n^{\#}(z) \leq n r^{n-1}$ AND IF $|z| \geq r > 1$, $f^{\#}(z) \leq n r^{-(n+1)}$

HENCE $f_n^{\#} \to 0$ COMPACTLY IN $\mathbb{D}$ OR $\mathbb{C} - \overline{\mathbb{D}}$, SO $\{f_n\}$ IS NORMAL.

BUT $f_n^{\#}(z) = \frac{n}{2} \to +\infty$ IF $|z| = 1$, SO $\{f_n\}$ CANNOT BE NORMAL.