LAST TIME:

WEIERSTRASS THM: IF $f_n \in \mathcal{O}(U)$ AND $f_n \to f$ COMPACTLY,
THEN $f \in \mathcal{O}(U)$ AND $f_n' \to f'$ COMPACTLY IN U.

COR: **WEIERSTRASS M-TEST:**
SPOZE $f_n \in \mathcal{O}(U)$ AND FOR EACH COMPACT $K \subset U$, THERE EXIST $M_n > 0$ SO

$$\sup_{z\in K} |f_n(z)| \leq M_n \quad \text{WITH } \sum_{n=1}^{\infty} M_n \text{ CONVERGING.}$$

THEN $\left(\sum_{n=1}^{\infty} f_n\right) \to f$ COMPACTLY IN U WITH $f \in \mathcal{O}(U)$

ALSO $\sum_{n=1}^{\infty} f_n' \to f'$ COMPACTLY IN U

Pf/ FIX $K \subset U$ (AND HENCE M_n), FOR $\varepsilon > 0$, WE FIND $N \geq 1$ SO THAT $\sum_{n=N}^{\infty} M_n < \varepsilon$. FOR EACH $z \in K$ AND ALL $m > n > N$

$$\left|\sum_{j=n}^{m} f_j(z)\right| \leq \sum_{j=n}^{m} |f_j(z)| \leq \sum_{j=n}^{m} M_j < \varepsilon$$

THAT IS, THE PARTIAL SUMS $\sum_{n=1}^{m} f_n(z)$ ARE LOCALLY UNIFORMLY CAUCHY SEQUENCES FOR EACH $z \in K$

SO $\sum_{n=1}^{\infty} f_n$ CONVERGES COMPACTLY TO f.

BY THM, $f \in \mathcal{O}(U)$ AND $f_n' \to f$ COMACTLY IN U.

WHAT PROPERTIES ARE CARRIED OVER
~~LEMMA:~~ WHEN WE PASS $f_n \to f$ COMPACTLY?

MOST BASICALLY, THE TOPOLOGICAL DEGREE (IE # OF PREIMAGES OF POINTS) IS PRESERVED:

LEMMA: LET $U \subset \mathbb{C}$ BE A DOMAIN WITH γ JORDAN CURVE IN U WITH $D = \mathrm{int}(\gamma) \subset U$, $f \in \mathcal{O}(U)$ WITH $f(z) \neq 0$ FOR $z \in \{\gamma\}$.
THEN THERE IS A $\delta > 0$ SO THAT

$$\sup_{z \in \{\gamma\}} |f(z) - g(z)| < \delta \Rightarrow N_f(D,0) = N_g(D,0)$$

Pf/ LET δ BE THE INFIMUM OF $|f(z)|$ ON $\{\gamma\}$. THEN $\delta > 0$.

IF $\sup_{z \in \{\gamma\}} |f(z) - g(z)| < \delta$, SINCE $|f(z)| \geq \delta$, WE CAN APPLY ROUCHÉS THM TO CONCLUDE THAT f AND g HAVE THE SAME NUMBER OF ZEROS IN D (WITH MULT) ∎

AN EASY EXAMPLE OF THIS IS POLYNOMIALS

$$P_t(z) = z^d + tz \quad (\text{FIXED } d)$$

P_0 HAS A ZERO OF ORDER d AT 0, AND FOR $t > 0$, WE GET THE $(d-1)^{ST}$ ROOTS OF t SURROUNDING 0.

HURWITZ THM: LET $U \subset \mathbb{C}$ BE A DOMAIN ON WHICH f_n IS HOLOMORPHIC FOR EACH n, AND $f_n \to f$ COMPACTLY IN U.
SPOZE FOR EACH COMPACT $K \subset U$, THERE IS $N_K \in \mathbb{Z}$ SO THAT
$f_n \neq 0$ ON K FOR $n \geq N$.
THEN EITHER $f \neq 0$ ON ALL OF U
OR f IS IDENTICALLY 0 ON U

SAID ANOTHER WAY,

COR. IF A SEQUENCE OF UNIVALENT FUNCTIONS f_n CONVERGES COMPACTLY IN U, TO f, THEN f IS UNIVALENT OR CONSTANT

DEF: A FUNCTION $f: U \to \mathbb{C}$ IS UNIVALENT ON U
IF IT IS HOLOMORPHIC AND ONE-TO-ONE ON U.
IN OTHER WORDS, IT IS A CONFORMAL MAP FROM U TO ITS IMAGE

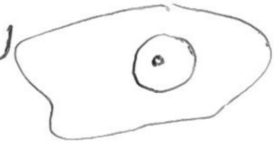
(SOMETIMES THE LEMMA IS CALLED HURWITZ'S THM)

BOTH CONCLUSIONS OF THM OCCUR,
CONSIDER $f_n(z) = e^z/n$

PF OF HURWITZ:

SPOZE f IS NOT IDENTICALLY 0, BUT $f(p)=0$ FOR SOME $p\in U$

THEN p IS ISOLATED, SO THERE IS $\mathbb{D}_r(p)$ WITH $\overline{\mathbb{D}_r(p)} \subset U$



AND $f(z) \neq 0$ ON $|z-p|=r$

SINCE $f_n \to f$ UNIFORMLY ON $\partial \mathbb{D}_r(p)$, THE LEMMA TELLS US THAT FOR n LARGE, f_n ALSO HAS A ZERO IN $\mathbb{D}_r(p)$ CONTRADICTING HYPOTHESIS. $\boxtimes$

PF OF COR: LET $p \neq q$, $p,q \in U$ WITH $f(p)=f(q)$.

LET $D = \mathbb{D}_r(q)$ WITH $p \notin D$ AND $\overline{D} \subset U$.

THEN $f_n - f_n(p)$ IS NON VANISHING IN D FOR EACH n (SINCE INJECTIVE)

AND $f_n - f_n(p) \to f - f(p)$ UNIFORMLY IN D.

BUT $f(q) - f(p) = 0$, SO $f = f(p)$ IDENTICALLY ON D AND HENCE ON ALL OF U. $\boxtimes$

ANOTHER

COR: LET $U \subset \mathbb{C}$ BE A DOMAIN WITH $f_n \in \mathcal{O}(U)$ AND $f_n \to f$ COMPACTLY IN U. SPOZE ALSO $f_n(U) \subset V$ WITH V OPEN.

THEN $f(U) \subset V$ OR f IS CONSTANT WITH $f(U) = p \in \partial V$

PF/ CERTAINLY $f(U) \subset \overline{V}$. SPOZE f TAKES ON THE VALUE $q \in \partial V$,

NOTE $f_n - q$ DOES NOT VANISH AND CONVERGES COMPACTLY TO $f - q$ IN U, BUT $f(z_0) - q = 0$ FOR SOME $z_0 \in U$. THUS BY HURWITZ THM, $f(z) \equiv q$ FOR ALL $z \in U$ $\blacksquare$

DEF A FAMILY $\mathcal{F} \subset \mathcal{C}(U, \mathbb{C})$ IS COMPACTLY BOUNDED IF FOR EVERY COMPACT $K \subset U$ THERE IS $M > 0$ SO THAT

$$z \in K \text{ AND } f \in \mathcal{F} \Rightarrow |f(z)| \leq M$$

THM: $\mathcal{F} \subset \mathcal{O}(U)$ IS PRECOMPACT $\iff$

(i) $\{f(p) \mid f \in \mathcal{F}\}$ IS A BOUNDED SUBSET OF $\mathbb{C}$ FOR ALL $p \in U$

(ii) THE FAMILY OF DERIVATIVES $\{f' \mid f \in \mathcal{F}\}$ IS COMPACTLY BOUNDED

IF U IS A DOMAIN (i) CAN BE RELAXED TO SOME $p \in U$

PF) $\Rightarrow$. (i) FOLLOWS IMMEDIATELY.

IF (ii) FAILS, THEN FOR SOME $K \subset U$ COMPACT, THERE ARE $z_n \in K$ AND $f_n \in \mathcal{F}$ WITH $|f_n'(z_n)| > n$ FOR $n \in \mathbb{Z}_+$

SINCE $\mathcal{F}$ IS PRECOMPACT, WE CAN PASS TO A SUBSEQUENCE $\{f_n\}$ CONVERGING COMPACTLY IN U.

BY THE WEIERSTRASS THM $\{f_n'\}$ CONVERGES COMPACTLY IN U AND SO UNIFORMLY ON K. THIS CONTRADICTS $|f_n'(z_n)| \to +\infty$.

$\Leftarrow$ SPOZE (i) AND (ii) HOLD. BY ARZELA-ASCOLI, WE ONLY NEED CHECK EQUICONTINUITY OF $\mathcal{F}$.

TAKE $p \in U$. THEN THERE IS r SO THAT $\overline{\mathbb{D}_r(p)} \subset U$.

BY (ii) WE HAVE $M > 0$ SO THAT $|f'(\zeta)| \leq M$ FOR $\zeta \in \mathbb{D}_r(p)$, $f \in \mathcal{F}$

HENCE
$$|f(z) - f(p)| = \left| \int_{[p,z]} f'(\zeta)\, d\zeta \right| \leq M|z - p|$$

FOR ALL $z \in \mathbb{D}_r(p)$ AND $f \in \mathcal{F}$.

THIS GIVES EQUICONTINUITY, SO $\mathcal{F}$ IS PRECOMPACT.

IF U IS A DOMAIN, WE ONLY NEED ONE $p \in U$ IN (i).

LET $z \in U$, AND USE PATH CONNECTIVITY OF U TO FIND A C^1 γ IN U WITH $\gamma(0) = p$ $\gamma(1) = z$.

THEN FIND $M_1, M_2 > 0$ WITH $|f(\zeta)| \leq M_1$, $|f'(\zeta)| \leq M_2$ FOR ALL $\zeta \in \{\gamma\}$ AND $f \in \mathcal{F}$ THEN

$$|f(z)| = \left| f(p) + \int_\gamma f'(\zeta)\,d\zeta \right| \leq M_1 + M_2 \,\mathrm{length}(\gamma)$$

FOR ALL $f \in \mathcal{F}$, SO f IS POINTWISE BOUNDED. ∎

SINCE ~~THE~~ A BOUND ON $f(z)$ FOR f HOLOMORPHIC GIVES A BOUND ON f', WE GET

MONTEL'S THM

$\mathcal{F} \subset \mathcal{O}(U)$ IS PRECOMPACT $\iff$ $\mathcal{F}$ IS COMPACTLY BOUNDED

(THIS IS USUALLY STATED AS

"A FAMILY $\mathcal{F}$ OF MEROMORPHIC FUNCTIONS ON A DOMAIN U ~~IS NORMAL~~ THAT OMITS THREE VALUES IN $\hat{\mathbb{C}}$ IS NORMAL"

A FAMILY $\mathcal{F} \subset \mathcal{M}(U)$ IS NORMAL IF IT IS PRECOMPACT IN $C(U, \hat{\mathbb{C}})$)

Proof of Montel's Thm

$\Rightarrow$/ Spoze $\mathcal{F}$ is not compactly bounded. Then there is a compact $K \subset U$ $z_n \in K$ and $\{f_n\} \subset \mathcal{F}$ with $|f_n(z_n)| > n$ for $n \in \mathbb{Z}^+$. No subseq. of $\{f_n\}$ can converge unif. on K, so $\mathcal{F}$ not precompact.

$\Leftarrow$/ Spoze $\mathcal{F}$ is compactly bounded, and take a compact $K \subset U$ and $r > 0$ ~~a closed r nbhd of K~~ so that

$$K_r = \{ z \in \mathbb{C} \mid \operatorname{dist}(z, K) \leq r \}$$

is contained in U.

Since $\mathcal{F}$ is bdd on K_r, $|f(z)| \leq M$ for $z \in K_r$ and $f \in \mathcal{F}$.

But then $\displaystyle\sup_{z \in K} |f'(z)| \leq \frac{M}{r}$ (since the image of $\mathbb{D}_r(z) \subseteq \mathbb{D}_M(0)$)

and hence $\mathcal{F}'$ is compactly bounded, so by the earlier thm $\mathcal{F}$ is precompact. $\blacksquare$

Important observation:

Idea of precompactness and Arzela-Ascoli theorem are topological, but Montel's thm is definitely a result depending on analiticity. For example

$f_n : [0, 2\pi] \to \mathbb{R}$ is uniformly bounded and C^∞-smooth,

but there are no uniformly convergent subsequences.

VITALY-PORTER THM

LET $U \subset \mathbb{C}$ BE A DOMAIN AND LET $E \subset U$ HAVE AN ACCUMULATION POINT IN U. SPOZE $f_n \in \mathcal{O}(U)$ FOR ALL n, WITH $\{f_n\}$ COMPACTLY BOUNDED, AND THAT FOR EACH $p \in E$, $\lim_{n\to\infty} f_n(p)$ EXISTS. THEN f_n CONVERGES COMPACTLY IN U.

Pf/ BY MONTEL'S THEOREM, $\{f_n\}$ IS PRECOMPACT, SO THERE IS A SUBSEQUENCE CONVERGING COMPACTLY TO $f \in \mathcal{O}(U)$. IN FACT $f_n \to f$ COMPACTLY.

FOR IF NOT, THERE IS A $K \subset U$ AND SOME $\varepsilon > 0$ FOR WHICH A SUBSEQUENCE f_{n_j} SATISFIES

$$\sup_{z\in K} |f_{n_j}(z) - f(z)| \geq \varepsilon$$

BUT THE SUBSEQUENCE $\{f_{n_j}\}$ IS ALSO PRECOMPACT, SO IT HAS A SUBSEQUENCE CONVERGING TO SOME $g \in \mathcal{O}(U)$.

THEN $\sup_{z\in K} |g(z) - f(z)| \geq \varepsilon$ AND SO $f \neq g$.

BUT SINCE $\{f_n\}$ CONVERGES POINTWISE ON E, $f(z) = g(z)$ FOR $z \in E$.

SINCE E HAS AN ACCUMULATION POINT IN U, WE HAVE $f = g$ ON U, A CONTRADICTION.