OUR GOAL IS TO PROVE THE RIEMANN MAPPING THEOREM (WHICH I'VE MENTIONED BEFORE):

RIEMANN MAPPING THEOREM

> IF U IS A SIMPLY CONNECTED DOMAIN IN $\mathbb{C}$ WITH $U \neq \mathbb{C}$, THEN U IS CONFORMALLY ISOMORPHIC TO $\mathbb{D}$

WE DO THIS BY FINDING A SEQUENCE f_n OF INJECTIVE HOLO FUNCTIONS MAPPING $U \to \mathbb{D}$ WITH $p \in U$ MAPPING TO 0. WE THEN MAXIMIZE $|f_n'(p)|$ AND SHOW THAT THIS IS THE DESIRED MAP.

BUT FIRST, WE NEED SOME MORE TOOLS, SPECIFICALLY ABOUT CONVERGENCE ~~OF~~ WITHIN FAMILIES OF HOLOMORPHIC FUNCTIONS.

LAST TIME, WE INTRODUCED ~~THE~~ A TOPOLOGY ON THE SET OF CONTINUOUS MAPS $f: U \to X$ WHERE $U \subset \mathbb{C}$ IS NONEMPTY AND OPEN AND X IS A METRIC SPACE:

THE COMPACT-CONVERGENCE TOPOLOGY, WHERE TWO MAPS f AND g ARE CLOSE ON A COMPACT SET $K \subset U$ IF FOR $\varepsilon > 0$,

$$\sup_{p \in K} \left(d(f(p), g(p)) \right) < \varepsilon .$$

THIS LEADS US TO ~~COMPACT~~ $f_n \to f$ COMPACTLY IN U IF EVERY BASIC NBHD OF f CONTAINS ALL BUT FINITELY MANY OF THE f_n.

A TOPOLOGICAL SPACE IS <u>PRECOMPACT</u> IF ITS CLOSURE IS COMPACT. (AKA "RELATIVELY COMPACT")

IN A METRIC SPACE, COMPACT $\Longleftrightarrow$ SEQUENTIALLY COMPACT, SO

<u>DEF</u> A FAMILY OF FUNCTIONS $\mathcal{F} \subset \mathcal{C}(U,X)$ IS <u>PRECOMPACT</u> IF IT HAS COMPACT CLOSURE.

EQUIVALENTLY, IF EVERY SEQUENCE $\{f_n\}$ WITH $f_n \in \mathcal{F}$ HAS A SUBSEQUENCE WHICH CONVERGES COMPACTLY IN U ~~WHEN~~ THEN $\mathcal{F}$ IS PRECOMPACT (NOTE THE LIMIT MAY NOT BE IN $\mathcal{F}$)

A RELATED NOTION:

<u>DEF:</u> THE FAMILY $\mathcal{F} \in \mathcal{C}(U,X)$ IS <u>EQUICONTINUOUS</u> IF, FOR EVERY $p \in U$ AND $\varepsilon > 0$, THERE IS A $\delta > 0$ SO THAT

$$(|p-q| < \delta \;\&\; f \in \mathcal{F}) \Rightarrow d(f(p), f(q)) < \varepsilon$$

THIS IS EQUIVALENT TO UNIFORM CONTINUITY ON COMPACT SUBSETS.

<u>LEMMA:</u> $\mathcal{F} \subset \mathcal{C}(U,X)$ IS EQUICONTINUOUS $\Longleftrightarrow$ FOR EVERY COMPACT $K \subset U$ AND $\varepsilon > 0$, THERE IS $\delta > 0$ SO THAT

$$(p \in K,\ |p-q| < \delta,\ f \in \mathcal{F}) \Rightarrow d(f(p), f(q)) < \varepsilon$$

Pf/ $\Leftarrow$ ~~is~~ JUST ~~THROUGH~~ THE DEF OF EQUICONTINUOUS.

FOR $\Rightarrow$, SPOZE $\mathcal{F}$ EQUICONTINUOUS BUT THE CONCLUSION FAILS. THEN THERE IS SOME COMPACT $K \subset U$ AND $\varepsilon > 0$, ~~SO THAT~~ AS WELL AS SEQUENCES $\{p_n\} \subset K$, $\{q_n\} \subset U$ AND $\{f_n\} \subset \mathcal{F}$ SO THAT

$$|p_n - q_n| < \frac{1}{n} \quad \text{BUT } d\big(f_n(p_n), f_n(q_n)\big) > 3\varepsilon$$

FOR ALL n.

PASS TO A SUBSEQUENCE SO WE HAVE $p_n \to p \in K$, AND HENCE $q_n \to p \in K$ ALSO.

SINCE $\mathcal{F}$ IS EQUICONTINUOUS, THERE IS A $\delta > 0$ SO THAT

$$|p_n - p| < \delta \Rightarrow d(f(p_n) - f(p)) < \varepsilon$$
$$|q_n - p| < \delta \Rightarrow d(f(q_n) - f(p)) < \varepsilon$$

FOR n LARGE ENOUGH.

BUT THEN $d(f(p_n), f(q_n)) < 2\varepsilon$, CONTRADICTION.

THIS GIVES US

ARZELA-ASCOLI THM: $\mathcal{F} \subset \mathcal{C}(U, X)$ IS PRECOMPACT

$\Leftrightarrow$ (i) FOR EVERY $p \in U$, THE SET $\{f(p) \mid f \in \mathcal{F}\}$ IS PRECOMPACT

& (ii) $\mathcal{F}$ IS EQUICONTINUOUS

ACTUALLY ADAPTS TO ANY LOCALLY COMPACT, σ-COMPACT, SEPARABLE HAUSDORFF U

(POINTS HAVE NBHDS W/ COMPACT CLOSURE) (U IS A COUNTABLE UNION OF COMPACT SETS) (HAS ~~a~~ COUNTABLE DENSE SUBSET)

Pf/ $\Rightarrow$

THE PRECOMPACTNESS OF $\mathcal{F}$ GIVES (i) IMMEDIATELY.

TO SEE (i) $\Rightarrow$ (ii), SPOZE (i) HOLDS BUT (ii) FAILS.

THEN THERE IS $p \in U$ AND $\varepsilon > 0$ WITH A SEQUENCE OF $q_n \in U$ $\{q_n\} \to p$ AND $\{f_n\} \in \mathcal{F}$ SUCH THAT $d(f_n(p), f_n(q_n)) > \varepsilon$ FOR ALL n

BUT SINCE $\mathcal{F}$ IS PRECOMPACT, $f_n \xrightarrow{\text{COMPACTLY}} f \in \mathcal{C}(U,X)$.

APPLY THE TRIANGLE INEQUALITY:

$$d(f_n(p), f_n(q_n)) \le \underbrace{d(f_n(p), f(p))}_{\to 0} + \underbrace{d(f(p), f(q_n))}_{\to 0 \text{ SINCE } f \text{ CONT.}} + \underbrace{d(f(q_n), f_n(q_n))}_{\to 0}$$

SINCE $f_n \to f$ UNIFORMLY ON NBHDS OF p WITH COMPACT CLOSURE.

SO WE HAVE A CONTRADICTION.

FOR $\Leftarrow$, $\mathcal{F}$ SATISFIES (i) AND (ii) AND LET $\{f_n\}$ BE ANY SEQ. IN $\mathcal{F}$.

LET E BE A COUNTABLE DENSE SUBSET OF U.

FOR EACH $p_k \in E$, f_n HAS A SUBSEQUENCE SO THAT $f_{n_j}(p_k)$ CONVERGES (SINCE $\mathcal{F}$ IS PRECOMPACT).

USING A DIAGONAL ARGUMENT SIMILAR TO CANTOR'S, WE FIND A SUBSEQUENCE $\{f_m\}$ OF $\{f_n\}$ ~~THAT CONVERGE SO THAT~~ THAT CONVERGES SIMILTANEOUSLY ON ALL $p_k \in E$.

NOW WE JUST NEED THE COMPACT CONVERGENCE OF $\{f_m\}$.

SO FIX $K \subset U$ AND $\varepsilon > 0$, AND BY THE LEMMA THERE IS $\delta > 0$ SO THAT FOR $p \in K$, $|p - q| < \delta \Rightarrow |f(p) - f(q)| < \varepsilon$

SINCE K IS COMPACT, WE CAN COVER IT BY FINITELY MANY DISKS $D_1, \dots, D_s$ OF RADIUS $\delta/2$. ~~CHOOSE P_j~~

FOR EACH DISK D_j CHOOSE $P_j \in E \cap D_j$ AND SINCE $\{f_m\}$ CONVERGES AT P_j THERE IS N SO THAT

$$d(f_k(P_j), f_\ell(P_j)) < \varepsilon \quad \text{FOR } k, \ell > N \text{ AND EACH } P_j$$

SINCE EACH POINT p OF E IS IN SOME j $|p - P_j| < \delta$, SO FOR

$$k, \ell > N \quad d(f_k(p), f_\ell(p)) \leq d(f_k(P_j), f_k(p)) + d(f_k(P_j), f_\ell(P_j)) + d(f_k(P_j), f_\ell(p))$$

$$\leq \varepsilon + \varepsilon + \varepsilon = 3\varepsilon$$

SO $\{f_m\}$ HAS A UNIFORM CAUCHY CONDITION AND HENCE CONVERGES UNIFORMLY ON K.

~~RETURNING~~

FOCUS NOW ON $\mathcal{O}(U) \subset \mathcal{C}(U, \mathbb{C})$

WITH THE EUCLIDEAN METRIC

$d(z,w) = |z-w|$

CONTRAST WITH THE SMOOTH FUNCTIONS IN $\mathcal{C}(U, \mathbb{C})$, WHERE FOR ANY CONTINUOUS $f \in \mathcal{C}(U, \mathbb{C})$ THERE IS A SEQUENCE OF POLYNOMIALS $P_n(x, y)$ $[x, y \in \mathbb{R}]$ WITH $P_n \to f$ COMPACTLY IN U (STONE-WEIERSTRASS THM)

THINGS ARE NICER IN $\mathbb{C}$.

THM (WEIERSTRASS) LET $f_n \in \mathcal{O}(U)$ AND $f_n \to f$ COMPACTLY IN U. THEN $f \in \mathcal{O}(U)$ AND $f_n' \to f'$ COMPACTLY IN U

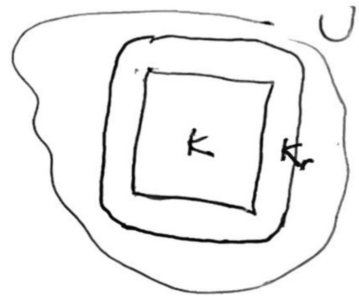
IN OTHER WORDS, THE SPACE $\mathcal{O}(U)$ IS CLOSED IN $\mathcal{C}(U, \mathbb{C})$ AND $D : f \mapsto f'$ IS CONTINUOUS FROM $\mathcal{O}(U) \to \mathcal{O}(U)$.

Pf/ SINCE THE f_n ARE CONTINUOUS, SO IS f.
LET γ BE A SMOOTH JORDAN CURVE IN U. SINCE $\{\gamma\}$ IS COMPACT, $f_n \to f$ ON γ UNIFORMLY. BY CAUCHY'S THM

$$\int_\gamma f(z)\,dz = \lim_{n\to\infty} \int_\gamma f_n(z)\,dz = 0$$

SO $f \in \mathcal{O}(U)$.

FOR $f' \to f$, FIX $\varepsilon > 0$ AND K COMPACT IN U. CHOOSE r SMALL ENOUGH THAT $K_r = \{ z \in \mathbb{C} \mid \operatorname{dist}(z, K) \le r \}$ IS IN U



~~THERE IS~~ SINCE $f_n \to f$, WE HAVE N SO THAT

$$\sup_{z \in K_r} |f_n(z) - f(z)| \le r\varepsilon \quad \text{FOR } n > N$$

AND FOR $z \in K$, $\overline{\mathbb{D}_r(z)} \subset K_r$.

SINCE $F_n = f_n - f$ MAPS ~~A~~ DISKS OF RADIUS $r\varepsilon$ INTO DISKS OF r,

$$\sup_{z \in K} |f_n'(z) - f'(z)| \le \frac{r\varepsilon}{r} = \varepsilon \qquad \text{FOR } n > N, \text{ SO } f_n' \to f' \text{ COMPACTLY}$$

$\square$

NOTE THAT $f_n \to f$ COMPACTLY DOESN'T MEAN THE CONVERGENCE ON OPEN SETS IS UNIFORM. FOR EXAMPLE,

LET f BE HOLOMORPHIC IN $\mathbb{D}$, SO $f(z) = \sum_{k=0}^{\infty} a_k z^k$.

LET f_n BE THE nth PARTIAL SUM, $f_n(z) = \sum_{k=0}^{n} a_k z^k$.

THEN $f_n \to f$ COMPACTLY IN $\mathbb{D}$

BUT IF $f = \frac{1}{1-z}$, $f_n = \frac{1 - z^{n+1}}{1-z}$

AND $\sup_{z \in \mathbb{D}} |f_n(z) - f(z)| = \sup_{z \in \mathbb{D}} \left| \frac{z^{n+1}}{1-z} \right| = +\infty$

Weierstrass M-test: Let $f \in \mathcal{O}(U)$ and for every compact $K \subset U$ there are constants $M_n \geq 0$ so

$$\sup_{z \in K} |f_n(z)| \leq M_n \quad \text{with} \quad \sum_{n=1}^{\infty} M_n < +\infty .$$

Then $\sum_{n=1}^{\infty} f_n$ converges compactly in U to $f \in \mathcal{O}(U)$

and $\sum_{n=1}^{\infty} f_n' \to f'$ compactly in U.

Pf/ Take $K \subset U$ and find M_n. For $\varepsilon > 0$, there is $N \geq 1$ so $\sum_{n=N}^{\infty} M_n < \varepsilon$, so for $z \in K$ and each $m > n > N$

$$\left| \sum_{j=n}^{m} f_j(z) \right| \leq \sum_{j=n}^{m} |f_j(z)| \leq \sum_{j=n}^{m} M_j < \varepsilon$$

In particular, the partial sums $\sum_{n=1}^{m} f_n(z)$ ~~are lo.~~ form locally uniform Cauchy sequences for $z \in K$ (uniform in z).

Thus $\sum_{n=1}^{\infty} f_n$ converges compactly in U to f.

By the prev. Weierstrass thm, $f \in \mathcal{O}(U)$ and $f_n' \to f$.