THE HYPERBOLIC METRIC ON $\mathbb{D}$

LET $\varphi : \mathbb{D} \to \mathbb{D}$. BE AN ISOMETRY. WE WANT TO FIND $g = \rho(z)|dz|$ WHICH IS INVARIANT UNDER ALL SUCH φ, IE $\rho = |\varphi'|(\rho\circ\varphi)$ FOR $\varphi \in \mathrm{Aut}(\mathbb{D})$

IF SUCH g EXISTS, THEN FOR $p \in \mathbb{D}$, $\rho(p) = |\varphi'(p)|\,\rho(0)$ IF $\varphi(p) = 0$.

BUT IF $\varphi \in \mathrm{Aut}(\mathbb{D})$, $|\varphi'(p)|$ DOESN'T DEPEND ON φ (ONLY ON p) BECAUSE ALL SUCH AUTOMORPHISMS ARE OF THE FORM

$$z \mapsto \alpha \frac{z-p}{1-\bar{p}z} \quad \text{WITH } |\alpha| = 1.$$

THUS $\rho(p)$ ONLY DEPENDS ON $\rho(0)$

THIS GIVES US $\rho(p) = \dfrac{\rho(0)}{1-|p|^2}$. CHOOSING $\rho(0) = 2$ GIVES

DEF: $g_{\mathbb{D}} = \dfrac{2}{1-|z|^2}|dz|$ IS THE $\left\{\begin{matrix}\text{HYPERBOLIC} \\ \text{POINCARÉ}\end{matrix}\right\}$ METRIC ON $\mathbb{D}$

THM: EVERY $f \in \mathrm{Aut}(\mathbb{D})$ IS A HYPERBOLIC ISOMETRY, IE $\|f'(z)\| = 1$ OR $|f'(z)| = \dfrac{1-|f(z)|^2}{1-|z|^2}$ FOR ALL $z \in \mathbb{D}$

EX: SINCE ANY BIHOLOMORPHISM $U \xrightarrow{f} V$ TRANSPORTS THE METRIC IN V TO ONE IN U, WE CAN WRITE THE POINCARE METRIC IN THE UPPER HALF-PLANE $\mathbb{H}$ FROM THE ONE IN $\mathbb{D}$ VIA THE CAYLEY MAP

$$w = \varphi(z) = \frac{i-z}{i+z},$$

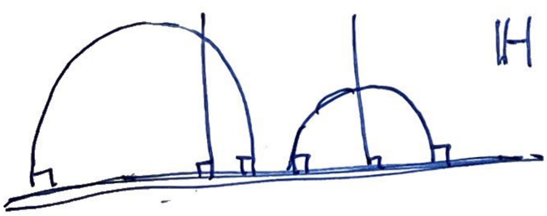
$$g_{\mathbb{H}} = \varphi^*\left(\frac{2}{1-|w|^2}|dw|\right) = \frac{2}{1-|\varphi(z)|^2}|\varphi'(z)||dz|$$

$$= \frac{2}{1-\left|\frac{i-z}{i+z}\right|^2}\left|\frac{2}{|i+z|^2}\right||dz|$$

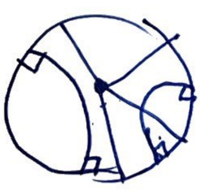
$$= \frac{4\,|dz|}{|i+z|^2-|i-z|^2} = \frac{|dz|}{\operatorname{Im}(z)}$$

HERE, IT IS EASY TO SEE THAT A UNIT TANGENT VECTOR IN $\mathbb{H}$ $x+iy$ HAS LENGTH y IN $g_{\mathbb{H}}$, SO IT SHRINKS TO 0 (~~IN~~ THE EUCLIDEAN NORM) AS $y \to 0$

OBVIOUSLY, IN $\mathbb{H}$ VERTICAL LINES ARE GEODESICS IN $g_{\mathbb{H}}$. SO ARE (EUCLIDEAN) CIRCLES $\perp$ TO THE REAL AXIS, BUT THAT TAKES A LITTLE WORK.



CONSEQUENTLY, PULLING BACK TO $\mathbb{D}$ WE GET GEODESICS AS ~~LINES~~ RADIAL LINES OR THE ~~UNIQUE~~ CIRCLES $\perp$ TO THE UNIT CIRCLE



THIS TAKES A LITTLE WORK... WE'LL DO THAT NEXT.

THM: TWO DISTINCT POINTS $p, q \in \mathbb{D}$ CAN BE JOINED BY A UNIQUE GEODESIC IN $g_{\mathbb{D}}$. THIS IS THE ARC OF THE EUCLIDEAN CIRCLE ORTHOGONGAL TO $\mathbb{T}$ AND PASSING THROUGH p AND q.

FURTHER, $$\mathrm{dist}_{\mathbb{D}}(p,q) = \log\left(\frac{|1-\bar{p}q| + |p-q|}{|1-\bar{p}q| - |p-q|}\right)$$

Pf/ THE BASIC IDEA IS THAT WE "KNOW" THE ANSWER IF $p, q \in \mathbb{R}$, AND THEN CAN MOVE THE GEODESICS AROUND BY AN ELEMENT OF $\mathrm{Aut}(\mathbb{D})$.

SO, LET $p=0$, $q=r$ WITH $0<r<1$,
AND LET $\gamma : [0,1] \to \mathbb{D}$ BE C^1 WITH $\gamma(0)=0$, $\gamma(1)=r$

TAKE $x(t) = \mathrm{Re}(\gamma(t))$ $\quad |x(t)|^2 \le |\gamma(t)|^2$

& $x'(t) \le |\gamma'(t)|$ FOR ALL t.

THEN
$$\mathrm{length}_{\mathbb{D}}(\gamma) = \int_0^1 \|\gamma'(t)\|_{\mathbb{D}}\, dt = \int_0^1 \frac{2|\gamma'(t)|}{1-|\gamma(t)|^2}\, dt$$
$$\geq \int_0^1 \frac{2x'(t)}{1-(x(t))^2}\, dt = \int_0^r \frac{2\,dx}{1-x^2} = \log\left(\frac{1+r}{1-r}\right)$$

WE HAVE EQUALITY $\iff$ $x(t) = |\gamma(t)|$ AND $x'(t) = |\gamma'(t)|$
ie WHEN $\gamma(t)$ IS THE STRAIGHT SEGMENT FROM 0 TO r.

A RIGID ROTATION IS AN ISOMETRY IN $\mathbb{D}$, SO

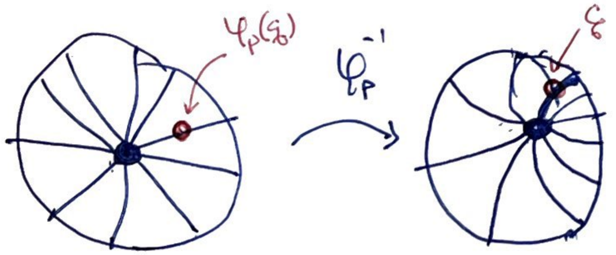
← GEODESICS

$$\mathrm{dist}_{\mathbb{D}}(0,z) = \log\left(\frac{1+|z|}{1-|z|}\right)$$

FOR ARBITRARY $p, q \in \mathbb{D}$, ~~...~~ WE CAN USE

$$\varphi_p(z) = \frac{z - p}{1 - \bar{p} z} \quad \text{TO SEND } p \to 0 \quad q \to \varphi_p(q) = \frac{q - p}{1 - \bar{p} q}$$

SO
$$\begin{aligned} \operatorname{dist}_{\mathbb{D}}(p, q) &= \operatorname{dist}_{\mathbb{D}}\left(0, \frac{q-p}{1-\bar{p}q}\right) \\ &= \log\left(\frac{|1-\bar{p}q| + |p-q|}{|1-\bar{p}q| - |p-q|}\right). \end{aligned}$$



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NOTE THAT φ_p SENDS RADIAL LINES TO CIRCLE SEGMENTS $\perp$ TO THE UNIT CIRCLE, SO THESE ARE ALSO GEODESICS IN $\mathbb{D}$ WITH THE POINCARÉ METRIC.

THIS ALSO TELLS US ABOUT WHICH PAIRS (p_1, p_2), (q_1, q_2) CAN BE MAPPED VIA AN ELEMENT OF $\mathrm{AUT}(\mathbb{D})$:

THM: LET (p_1, p_2) AND (q_1, q_2) BE PAIRS OF DISTINCT POINTS IN $\mathbb{D}$. THEN THERE IS A UNIQUE ELEMENT φ OF $\mathrm{AUT}(\mathbb{D})$ ~~SENDING~~ WITH $\varphi(p_1) = q_1$, $\varphi(p_2) = q_2$ $\iff$ $\operatorname{dist}_{\mathbb{D}}(p_1, p_2) = \operatorname{dist}_{\mathbb{D}}(q_1, q_2)$.

Pf/ $\Leftarrow$ IS IMMEDIATE: ANY $\varphi \in \mathrm{Aut}(\mathbb{D})$ IS AN ISOMETRY IN THE HYP. METRIC.

$\Rightarrow$ SPOZE $\mathrm{dist}_{\mathbb{D}}(p_1, p_2) = \mathrm{dist}_{\mathbb{D}}(q_1, q_2)$.

~~SEE~~ TAKE $\varphi \in \mathrm{Aut}(\mathbb{D})$ WITH $\varphi(p_1) = 0$, $\varphi(p_2) = p$.

$\psi \in \mathrm{Aut}(\mathbb{D})$ WITH $\psi(q_1) = 0$, $\psi(q_2) = q$.

SINCE φ, ψ ARE ISOMETRIES $\mathrm{dist}_{\mathbb{D}}(p_1, p_2) = |p|$

$\mathrm{dist}_{\mathbb{D}}(q_1, q_2) = |q|$

AND $|p| = |q|$. NOW $\alpha : z \mapsto \frac{q}{p} z$ IS A RIGID ROTATION

SO $\psi^{-1} \circ \alpha \circ \varphi$ MAPS (p_1, p_2) TO (q_1, q_2).

<u>THM</u> (PICK'S THM): EVERY HOLOMORPHIC MAP $f : \mathbb{D} \to \mathbb{D}$ IS EITHER AN ISOMETRY OR CONTRACTS THE HYPERBOLIC METRIC. IE

$$\|f'(z)\| \leq 1 \quad \text{OR} \quad |f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2} \quad \text{FOR ALL } z \in \mathbb{D}.$$

~~LEMMA~~ EQUALITY HOLDS AT SOME $z \in \mathbb{D}$

$\Longleftrightarrow f \in \mathrm{Aut}(\mathbb{D})$

EQUIVALENTLY IF $f \in \mathcal{O}(\mathbb{D})$ ~~THEN~~ AND $p, q \in \mathbb{D}$

$$\mathrm{dist}_{\mathbb{D}}(f(p), f(q)) \leq \mathrm{dist}_{\mathbb{D}}(p, q).$$

~~IF $\exists$ $p, q \in \mathbb{D}$ WITH $p \neq q$, ...~~ WITH EQUALITY HOLDING FOR ~~A~~ PAIR $p \neq q$

$\Longleftrightarrow f \in \mathrm{Aut}(\mathbb{D})$

THIS IS REALLY JUST THE SCHWARZ LEMMA INTERPRETED IN THE HYPERBOLIC METRIC.

Pf/ LET $p \in \mathbb{D}$, $a \in f(p)$.

THEN $h = \varphi_a \circ f \circ \varphi_{-p}$ HAS $h(0) = 0$, $h : \mathbb{D} \to \mathbb{D}$.

SO BY SCHWARZ LEMMA, $|h'(0)| \leq 1$.

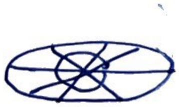
BUT $f = \varphi_{-a} \circ h \circ \varphi_p$, SO

$$\|f'(p)\| = \underbrace{\|\varphi_a'(0)\|}_{\text{ISOMETRIES}} \cdot \|h'(0)\| \, \underbrace{\|\varphi_p'(0)\|}_{\text{ISOMETRIES}} = \|h'(0)\| = |h'(0)| \leq 1$$

ALSO BY SCHWARZ, $|h'(0)| = 1 \iff h$ IS A RIGID ROTATION.

IN 1938, AHLFORS SHOWED THE SCHWARZ-PICK LEMMA IS REALLY ABOUT THE NEGATIVE CURVATURE OF THE METRIC.

SUMMARY: GEOMETRY $\iff$ METRIC

GEOMETRY	EUCLIDEAN	SPHERICAL	HYPERBOLIC
INFINITESIMAL LENGTH	$\lvert dz \rvert$	$\dfrac{2\lvert dz\rvert}{1+\lvert z\rvert^2}$	$\dfrac{2\lvert dz\rvert}{1-\lvert z\rvert^2}$
ORIENTATION PRESERVING ISOMETRIES	$z \mapsto \alpha z + b$, $\lvert\alpha\rvert = 1$	ROTATIONS OF $\hat{\mathbb{C}}$	$z \mapsto \dfrac{z-p}{1-\bar{p}z}$ $(p \in \mathbb{D})$
CURVATURE	0	+1	−1
GEODESICS	LINES	GREAT CIRCLES	CIRCLES $\perp$ UNIT CIRC.
SUM OF ANGLES IN A TRIANGLE (FROM GAUSS-BONNET)	π	$>\pi$	$<\pi$
CIRCUMFERENCE	$2\pi r$	$2\pi r - \dfrac{\pi r^3}{3} + O(r^5)$	$2\pi r + \dfrac{\pi r^3}{3} + O(r^5)$
DISKS			

NOW WE TURN TO CONVERGENCE IN FAMILIES OF FUNCTIONS. TO DO THIS, WE NEED AN APPROPRIATE TOPOLOGY.

TAKE $U \subset \mathbb{C}$ NONEMPTY, OPEN, AND X A METRIC SPACE WITH DISTANCE d
(USUALLY FOR US $X = \mathbb{C}$ OR $\hat{\mathbb{C}}$)

LET $\mathcal{C}(U, X) = \{ f : U \to X \mid f \text{ CONTINUOUS} \}$

WE USE THE COMPACT-OPEN TOPOLOGY / COMPACT CONVERGENCE TOP

DEF: A BASIC NEIGHBORHOOD OF $f \in \mathcal{C}(U,X)$ IS AN OPEN BALL

$$B(f, K, \varepsilon) = \{ g \in \mathcal{C}(U,X) \mid \sup_{p \in K} d(f(p), g(p)) < \varepsilon \}$$

WHERE $K \subset U$ IS COMPACT, $\varepsilon > 0$.

A SUBSET OF $\mathcal{C}(U,X)$ IS OPEN $\iff$ IT IS THE UNION OF BASIC NEIGHBORHOODS.

CONVERGENCE IN $\mathcal{C}(U,X)$ DEPENDS ONLY ON THE TOPOLOGY OF X AND NOT ON THE SPECIFIC METRIC d:

THE CONDITION $d(f(p), g(p)) < \varepsilon$ CAN BE REPLACED BY HAVING THE PAIR $(f(p), g(p))$ BELONGING TO A NEIGHBORHOOD OF THE DIAGONAL IN $X \times X$.