METRICS

A RIEMANNIAN METRIC g IN U MEASURES LENGTHS (AND ANGLES) OF TANGENT VECTORS AT POINTS OF U. THAT IS, AT EACH POINT $p \in U$ WE HAVE THE TANGENT SPACE $T_p U$ CONSISTING OF ALL VECTORS WITH INITIAL POINT p $(\cong \mathbb{R}^n)$.

g ASSIGNS A NORM $\|\cdot\|_g$ TO THESE SO THAT

- $\|v\|_g \geq 0$ FOR EVERY $v \in T_p U$ WITH $\|v\|_g = 0 \Leftrightarrow v = 0$
- $\|tv\|_g = |t| \|v\|_g$ FOR ALL $v \in T_p U$, ALL $t \in \mathbb{R}$
- $\|v_1 + v_2\|_g \leq \|v_1\|_g + \|v_2\|_g$ FOR ALL $v_1, v_2 \in g$
- $\|\cdot\|$ IS CONTINUOUS: FOR ANY CONTINUOUS VECTOR FIELD $p \mapsto v(p)$ ON U $p \mapsto \|v(p)\|_g$ IS CONT.

DEF A METRIC g IS A <u>CONFORMAL METRIC</u> IF THERE IS A POSITIVE CONTINUOUS FUNCTION $\rho : U \to \mathbb{R}$ SO THAT $\|v\|_g = \rho(p)|v|$ FOR ALL $p \in U$, $v \in T_p U$.

ρ IS THE <u>DENSITY FUNCTION</u> OF g.

- g IS C^r-SMOOTH IF ρ IS.
- THE <u>EUCLIDEAN NORM</u> IS JUST $\rho \equiv 1$

OFTEN WRITE $g = \rho(z)|dz|$

$$\text{length}_g(\gamma) = \int_a^b \|\gamma'(t)\|_g \, dt = \int_\gamma \rho(z)|dz|$$

THIS ALLOWS US TO DEFINE DISTANCE WRT g:

$$\mathrm{dist}_g(p,q) = \inf_{\gamma} \mathrm{length}_g(\gamma)$$

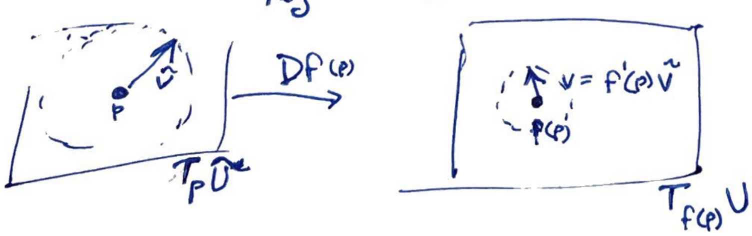
WHERE γ RANGES OVER PIECEWISE C^1 CURVES IN U FROM p TO q.

DEF A PIECEWISE C^1 CURVE $\gamma : [a,b] \to U$ IS A MINIMAL GEODESIC IF $\mathrm{length}_g(\gamma) = \mathrm{dist}_g(\gamma(a), \gamma(b))$

NEARBY POINTS CAN BE JOINED BY UNIQUE MINIMAL GEODESICS, BUT FOR ARBITRARY PAIRS, UNIQUENESS (OR EVEN EXISTENCE) MAY FAIL.

GIVEN DOMAINS $\tilde{U}$, U WITH $f : \tilde{U} \to U$, $f' \neq 0$ ON $\tilde{U}$ AND A CONF. METRIC g ON U, WE CAN DEFINE THE PULLBACK METRIC f^*g ON $\tilde{U}$ BY

$$\|\tilde{v}\|_{f^*g} = \rho(f(p))\,|f'(p)|\,\|v\|_g$$



WE CAN EXTEND THIS TO ANY NON-CONSTANT HOLOMORPHIC FUNCTION f BY VIEWING f^*g AS A SINGULAR METRIC IN $\tilde{U}$, VANISHING AT CRITICAL POINTS OF f.

THIS GIVES US A <u>DERIVATIVE NORM</u> $\|f'(p)\|$ FOR $f: \tilde{U} \to U$

$$\|f'(p)\| = \frac{\|v\|_g}{\|\tilde{v}\|_{\tilde{g}}} = \frac{\rho(f(p))\,|f'(p)|}{\tilde{\rho}(p)}$$

NOTE THIS DOESN'T DEPEND ON THE CHOICE OF v, SINCE THE METRICS g AN $\tilde{g}$ ARE CONFORMAL.

$$f:(\tilde{U},\tilde{g}) \to (U,g) \quad \text{IS A} \begin{Bmatrix} \text{CONTRACTION} \\ \text{LOCAL ISOMETRY} \\ \text{EXPANSION} \end{Bmatrix} \text{AT } z \text{ IF } \begin{Bmatrix} \|f'(z)\| < 1 \\ = 1 \\ > 1 \end{Bmatrix}$$

NOTE THAT THIS DEPENDS NOT ONLY ON f BUT ON $\tilde{g}$ AND g AS WELL.

IF WE HAVE

$$(U_1, g_1) \xrightarrow{f_1} (U_2, g_2) \xrightarrow{f_2} (U_3, g_3)$$

THEN BY THE CHAIN RULE

$$\|(f_2 \circ f_1)'(z)\|_{g_1 g_3} = \|f_2'(f_1(z))\|_{g_2 g_3} \cdot \|f_1'(z)\|_{g_1, g_2}$$

SINCE $\text{length}_g(f \circ \gamma) = \int_{f\circ\gamma} \rho(w)\,|dw| = \int_\gamma \rho(f(z))\,|f'(z)|\,|dz|$

$$= \int_\gamma \|f'(z)\|\,\tilde{\rho}(z)\,|dz|$$

~~WE HAVE~~

LEMMA SPOZE $(\tilde{U},\tilde{g}) \xrightarrow{f} (U,g)$ IS HOLOMORPHIC

(i) FOR EVERY PIECEWISE C^1 γ IN $\tilde{U}$, $\text{length}(f\circ\gamma) \le \sup_{z\in\gamma} \|f'(z)\|\, \text{length}_{\tilde{g}}(\gamma)$

(ii) FOR $p, q \in \tilde{U}$ $\quad \text{dist}(f(p), f(q)) \le \sup_{z\in\tilde{U}} \|f'(z)\|\, \text{dist}_{\tilde{g}}(p,q)$

CHARACTERIZATION OF LOCAL ISOMETRIES.

THM LET $f:(\tilde{U},\tilde{g})\to(U,g)$ BE NON-CONSTANT.

THE FOLLOWING ARE EQUIVALENT:

(i) $\|f'(z)\|=1$ FOR ALL $z\in\tilde{U}$

(ii) $f^*g=\tilde{g}$

(iii) $\rho(f(z))\,|f'(z)|=\tilde{\rho}(z)$ FOR ALL $z\in\tilde{U}$

(iv) $\mathrm{length}_g(f\circ\gamma)=\mathrm{length}_{\tilde{g}}(\gamma)$ FOR ALL PIECEWISE C^1 $\gamma\in\tilde{U}$

Pf) (i)$\iff$(ii)$\iff$(iii) IMMEDIATELY FROM DEF,

(i) $\Rightarrow$ (iv) FROM PREV. LEMMA.

(iv) $\Rightarrow$ (i):

IF (iv), THEN $\displaystyle\int_\gamma \|f'(z)\|\,\tilde{\rho}(z)\,|dz| = \int_\gamma \tilde{\rho}(z)\,|dz|$ FOR ALL $\gamma\subset\tilde{U}$

FOR SUFF SMALL $\varepsilon>0$, TAKE $\gamma=p+t$ WITH $t\in[0,\varepsilon]$.

THEN
$$\frac{1}{\varepsilon}\int_0^\varepsilon \|f'(p+t)\|\,\tilde{\rho}(p+t)\,dt=\frac{1}{\varepsilon}\int_0^\varepsilon\tilde{\rho}(p+t)\,dt$$

LET $\varepsilon\to0$, AND $\|f'(p)\|=1$ BY CONTINUITY.

ALSO

(v) FOR ANY $p\in\tilde{U}$, THERE IS A NBHD D OF p SO THAT
$$\mathrm{dist}_g(f(p),f(q))=\mathrm{dist}_{\tilde{g}}(p,q)\quad\text{FOR ALL } p,q\in D$$

NOTE THAT LOCAL ISOMETRIES NEED NOT BE GLOBAL ONES.

EXAMPLE: CONSIDER $g = \frac{-|dw|}{|w| \log|w|}$ ON $\mathbb{D}^*(0)$

AND $f(z) = z^2$. THEN

$$f^* g = -\frac{2|dz|}{|z^2| \log|z^2|} = \frac{-|dz|}{|z| \log|z|} = g$$

f IS A LOCAL ISOMETRY, BUT NOT GLOBAL SINCE NOT INJECTIVE, IE

$$\mathrm{dist}\left(\tfrac{1}{2}, \tfrac{-1}{2}\right) = 0 .$$

COR: IF $f : (\tilde{U}, \tilde{g}) \to (U, g)$ IS A LOCAL ISOMETRY AND MAPS $\tilde{U} \to U$ BIHOLOMORPHICALLY, THEN f IS A GLOBAL ISOMETRY, IE $\mathrm{dist}(f(p), f(q)) = \mathrm{dist}(p,q)$ FOR ALL $p, q \in \tilde{U}$

WE CAN EXTEND THIS TO CONF. METRICS ON $\hat{\mathbb{C}}$ BY TAKING CARE AT ∞; WE REQUIRE THAT THE PULLBACK OF g UNDER $z \mapsto 1/z$ BE WELL DEFINED NEAR 0.

THAT IS $\lim\limits_{z \to 0} \frac{1}{|z^2|} \rho(1/z)$ NEEDS TO EXIST AND BE POSITIVE.

EX: THE SPHERICAL METRIC $\sigma = \frac{2|dz|}{1+|z|^2}$

IS WELL DEFINED ON $\hat{\mathbb{C}}$

JUST CHECK $\frac{1}{|z^2|} \rho\left(\frac{1}{z}\right) = \rho(z)$ FOR $z \neq 0$.

THE HYPERBOLIC METRIC ON $\mathbb{D}$

LET $\varphi : \mathbb{D} \to \mathbb{D}$. BE AN ISOMETRY. WE WANT TO FIND $g = \rho(z)|dz|$ WHICH IS INVARIANT UNDER ALL SUCH φ, IE $\rho = |\varphi'|(\rho \circ \varphi)$ FOR $\varphi \in Aut(\mathbb{D})$

IF SUCH g EXISTS, THEN FOR $p \in \mathbb{D}$, $\rho(p) = |\varphi'(p)|\rho(0)$

IF $\varphi(p) = 0$.

BUT IF $\varphi \in Aut(\mathbb{D})$, $|\varphi'(p)|$ DOESN'T DEPEND ON φ (ONLY ON p) BECAUSE ALL SUCH AUTOMORPHISMS ARE OF THE FORM

$$z \mapsto \alpha \frac{z - p}{1 - \bar{p}z} \quad \text{WITH } |\alpha| = 1.$$

THUS $\rho(p)$ ONLY DEPENDS ON $\rho(0)$

THIS GIVES US $\rho(p) = \dfrac{\rho(0)}{1 - |p|^2}$. CHOOSING $\rho(0) = 2$ GIVES

DEF: $g_{\mathbb{D}} = \dfrac{2}{1 - |z|^2}|dz|$ IS THE $\left\{ \begin{matrix} \text{HYPERBOLIC} \\ \text{POINCARÉ} \end{matrix} \right\}$ METRIC ON $\mathbb{D}$

THM: EVERY $f \in Aut(\mathbb{D})$ IS A HYPERBOLIC ISOMETRY, IE $\|f'(z)\| = 1$ OR $|f'(z)| = \dfrac{1 - |f(z)|}{1 - |z|^2}$

FOR ALL $z \in \mathbb{D}$