LAST TIME WE SAW THAT THE GROUP OF MÖBIUS TRANSFORMATIONS (IE LINEAR FRACTIONAL TRANSF.)

$$\left\{ f:\hat{\mathbb{C}}\to\hat{\mathbb{C}} \;\middle|\; f(z)=\frac{az+b}{cz+d},\ \begin{matrix} a,b,c,d\in\mathbb{C} \\ ad-bc\neq 0\end{matrix} \right\} = \text{MÖB}$$

ARE BIHOLOMORPHISMS OF $\hat{\mathbb{C}}$, ISOMORPHIC TO

$$\text{MÖB} \cong PSL_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a,b,c,d\in\mathbb{C}\ \ ad-bc=1 \right\} \Big/ \{\pm \mathrm{Id}\}$$

AND GENERATED BY

- TRANSLATIONS : $z \mapsto z+\beta$
- LINEAR / DILATION : $z \mapsto \alpha z$
- INVERSION : $z \mapsto 1/z$.

PRESERVING CIRCLES IN $\hat{\mathbb{C}}$.

DEF: GIVEN $U\subset\hat{\mathbb{C}}$, THE <u>AUTOMORPHISM GROUP</u> $\mathrm{Aut}(U)$ IS THE GROUP OF BIHOLOMORPHISMS $U\to U$ UNDER COMPOSITION

THM $\boxed{\mathrm{Aut}(\hat{\mathbb{C}}) = \text{MÖB}}$

PF; SINCE ANY $\mu\in$ MÖB IS A BIHOLOMORPHISM, $\mu\in\mathrm{Aut}(\hat{\mathbb{C}})$.
GIVEN $f\in\mathrm{Aut}(\hat{\mathbb{C}})$, LETS SHOW $f\in$ MÖB.
SINCE MÖB IS 3-TRANSITIVE ON $\hat{\mathbb{C}}$, THERE IS A $\mu\in$ MÖB ~~with~~ WITH $\mu(f(\infty))=\infty$
THAT $h=\mu\circ f$ FIXES ∞.

SINCE h IS A BIHOLOMORPHISM, $h^{-1}(\infty)=\infty$
THUS, h IS A DEGREE 1 POLYNOMIAL, IE $h(z)=\alpha z+\beta$. PB

COR: $\mathrm{AUT}(\mathbb{C}) = \{ f \in \text{MÖB} \mid f(z) = \alpha z + \beta,\ \alpha \in \mathbb{C}^*,\ \beta \in \mathbb{C} \}$.
= AFFINE MAPS.

Pf/ CERTAINLY ANY AFFINE MAP IS IN $\mathrm{AUT}(\mathbb{C})$.
GIVEN $f \in \mathrm{AUT}(\mathbb{C})$, EXTEND IT TO A HOMEO $\hat{f}$ OF $\hat{\mathbb{C}}$
BY SETTING $\hat{f}(\infty) = \infty$.
THE SINGULARITY AT ∞ IS REMOVABLE, SINCE THE MAP
$g(z) = \dfrac{1}{f(1/z)}$ IS HOLOMORPHIC IN $\mathbb{D}_r^*(0)$ FOR r SMALL,
AND $\lim_{z \to 0} g(0) = 0$. SO $\hat{f} \in \mathrm{AUT}(\hat{\mathbb{C}})$ AND
SO IS MÖBIUS. SINCE $\hat{f}(\infty) = \infty$, IT IS AFFINE.

COR: THE ACTION OF $\mathrm{AUT}(\mathbb{C})$ IS SIMPLY 2-TRANSITIVE
IE FOR ANY TWO PAIRS OF POINTS (p_1, p_2), (q_1, q_2),
THERE IS A UNIQUE ELEMENT μ OF $\mathrm{AUT}(\mathbb{C})$ WITH $\mu(p_1) = q_1$, $\mu(p_2) = q_2$

THIS FOLLOWS FROM THE 3-TRANSITIVITY OF $\mathrm{AUT}(\hat{\mathbb{C}})$.
$(p_1, p_2, \infty) \mapsto (q_1, q_2, \infty)$.

THM $\mathrm{AUT}(\mathbb{D})$ IS THE SUBGROUP OF MÖB
$$\left\{ z \mapsto \alpha \frac{z - p}{1 - \bar{p} z} \;\middle|\; |\alpha| = 1,\ |p| < 1 \right\}$$

- SPECIFICALLY, AN AUTOMORPHISM OF $\mathbb{D}$ SEND THE UNIT CIRCLE TO ITSELF AND PRESERVES ORIENTATION.
- IF AN AUTOMORPHISM OF $\mathbb{D}$ FIXES THE ORIGIN, IT IS A RIGID ROTATION $z \mapsto \alpha z$ WITH $\alpha \in \mathbb{T}$

TO PROVE THIS, WE NEED THE FOLLOWING IMPORTANT RESULT:

THE SCHWARZ LEMMA:

> LET $f : \mathbb{D} \to \mathbb{D}$ BE HOLOMORPHIC WITH $f(0) = 0$
>
> THEN $|f(z)| \leq |z|$ FOR ALL $z \in \mathbb{D}$
>
> WITH $|f'(0)| \leq 1$.
>
> IF $|f(z_0)| = |z_0|$ FOR SOME $z_0 \in \mathbb{D}^*$, OR IF $|f'(0)| = 1$
>
> THEN f IS A RIGID ROTATION, IE $f(z) = \alpha z$ WITH $|\alpha| = 1$

(WE'VE ALREADY SEEN THAT $f : \mathbb{D} \to \mathbb{D}$ HOLO $\Rightarrow |f'(0)| \leq 1$)

PF/ WRITE $f(z) = \sum_{n=1}^{\infty} a_n z^n$ $(a_0 = 0$ SINCE $f(0) = 0)$.

$g(z) = \frac{f(z)}{z} = \sum_{n=0}^{\infty} a_{n+1} z^n$ HAS A REMOVABLE SINGULARITY AT 0,

SO $g \in \mathcal{O}(\mathbb{D})$.

IF $|z| = r$ WITH $0 < r < 1$, THEN $|g(z)| = \left|\frac{f(z)}{z}\right| < \frac{1}{r}$,

SO BY THE MAXIMUM PRINCIPLE $|z| < r \Rightarrow |g(z)| < 1/r$

THUS $\lim |g(z)| \leq 1$, SO $|g(z)| \leq 1$ FOR ALL z,

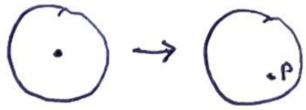
IE $|f(z)| \leq |z|$

AND $|f'(0)| = \lim_{z \to 0} |g(z)| \leq 1$.

IF $|f(z_0)| = |z_0|$ FOR SOME $z_0 \in \mathbb{D}^*$ (OR IF $|f'(0)| = 1$),

THEN $|g(z_0)| = 1$ AND BY THE MAXIMUM PRINCIPLE,

$g(z) = \alpha$ WITH $|\alpha| = 1$. THUS $f(z) = \alpha z$.



LETS CONSIDER $\varphi_p(z) = \dfrac{z - p}{1 - \bar{p} z}$ WITH $p \in \mathbb{C}$, $|p| \neq 1$

NOTE:

(i) $\varphi_p(0) = -p$, $\varphi_p(p) = 0$

(ii) $\varphi_p'(z) = \dfrac{(1-\bar{p}z)(1) - (z-p)(-\bar{p})}{(1-\bar{p}z)^2} = \dfrac{1-|p|^2}{(1-\bar{p}z)^2}$

SO $\varphi_p'(0) = 1 - |p|^2$ $\qquad \varphi_p'(p) = \dfrac{1}{1-|p|^2}$

(iii) $\varphi_p^{-1}(z) = \varphi_{-p}(z) = \dfrac{z+p}{1+\bar{p}z}$

(iv) IF $|p|<1$, $\varphi_p \in \mathrm{Aut}(\mathbb{D})$.

~~THE~~ (i) – (iii) ARE JUST CALCULATIONS. FOR (iv) NOTE

IF $|z|=1$, $|\varphi_p(z)| = \left|\dfrac{z-p}{1-\bar{p}z}\right| \overset{\text{SINCE } |z|=1}{=} \dfrac{|z-p|}{|z|\,|\bar{z}-\bar{p}|} = \dfrac{1}{|z|}\cdot 1 = 1$

SO USING THIS, WE CAN PROVE

$$\mathrm{Aut}(\mathbb{D}) = \left\{ z \mapsto \alpha \frac{z-p}{1-\bar{p}z} \;\middle|\; |p|<1,\ |\alpha|=1 \right\}.$$

Pf/ CERTAINLY $\alpha \varphi_p(z) \in \mathrm{Aut}(\mathbb{D})$.

GIVEN $f \in \mathrm{Aut}(\mathbb{D})$, LET $p = f^{-1}(0)$, AND LET

$g = f \circ \varphi_{-p} \in \mathrm{Aut}(\mathbb{D})$ WITH $g(0) = 0$. ~~ALSO $g^{-1} \in \mathrm{Aut}(\mathbb{D})$~~.

BY THE SCHWARZ LEMMA, $|g'(0)| \leq 1$.

BUT $g^{-1} \in \mathrm{Aut}(\mathbb{D})$ WITH $g^{-1}(0) = 0$, SO $|(g^{-1})'(0)| \leq 1$, ie $1 \leq |g'(0)|$

SO $|g'(0)| = 1$ AND g IS A RIGID ROTATION.

HENCE $f(z) = \alpha \dfrac{z-p}{1-\bar{p}z}$ ⊠

REMARK: ~~[scribbled out]~~

~~But~~ $\mathrm{Aut}(\mathbb{D})$ is not 2-transitive, since if $f \in \mathrm{Aut}(\mathbb{D})$ fixes the origin, it is a rigid rotation so, eg, $(0,p) \to (0,q)$ is not possible unless $|p| = |q|$. The question of other pairs $(p_1, p_2) \to (q_1, q_2)$ will be answered later.

NOTE THAT SINCE THE CAYLEY MAP

$$\varphi(z) = \frac{i - z}{i + z}$$

IS A BIHOLOMORPISM FROM THE UPPER-HALF PLANE $\mathbb{H}$ TO THE UNIT DISK $\mathbb{D}$ $(\varphi(0)=1,\ \varphi(i)=0,\ \varphi(\infty)=-1)$

$$\mathrm{Aut}(\mathbb{D}) \cong \mathrm{Aut}(\mathbb{H}).$$

BUT IT IS USEFUL TO BE EXPLICITE:

THM: $\mathrm{Aut}(\mathbb{H})$ IS THE SET OF MÖBIUS MAPS

$$z \mapsto \frac{az+b}{cz+d} \quad \text{with} \quad ad-bc>0, \quad a,b,c,d \in \mathbb{R}$$

$$\boxed{\mathrm{Aut}(\mathbb{H}) \cong PSL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, ad-bc=1,\ a,b,c,d \in \mathbb{R} \right\} \Big/ \{\pm \mathrm{Id}\}}$$

CONJUGACIES

DEF TWO MAPS ~~f & g~~ $f : U_f \to V_f$ AND $g : U_g \to V_g$ ARE CONJUGATE IF THERE IS A ONE-TO-ONE MAP ~~h~~ $h : (U_f \cup V_f) \to (U_g \cup V_g)$ SO THAT $g = h \circ f \circ h^{-1}$

$$\begin{tikzcd} U_f \arrow[r, "f"] \arrow[d, "\varphi"'] & V_f \arrow[d, "\varphi"] \\ U_g \arrow[r] & V_g \end{tikzcd}$$

THAT IS f & g ARE "THE SAME" WITH THE CHANGE OF VARIABLE $w = \varphi(z)$.

- NOTE THAT CONJUGACY IS AN EQUIVALENCE RELATION, PRESERVED UNDER COMPOSITION (ie $\underbrace{f \circ f \circ \cdots \circ f}_{n\text{ TIMES}} = f^{\circ n} \sim g^{\circ n} \Leftrightarrow f \sim g$)

DEF p IS A FIXED POINT OF f IF $f(p) = p$

THE MULTIPLIER OF f AT p IS $f'(p)$

NOTE THAT IF f IS CONJUGATE TO g BY φ AND p IS A FIXED POINT OF f, WITH $\varphi(p) = q$

$$f'(p) = g'(q)$$

BY THE CHAIN RULE (SINCE $\varphi'(p) \neq 0$).

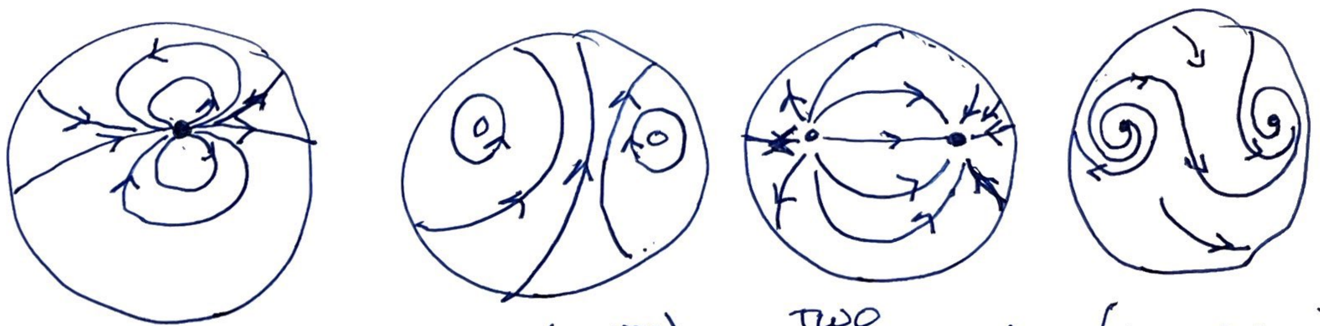
- IF $f \in$ MÖB WITH $f \neq id$, THEN f HAS EITHER 1 OR 2 FIXED POINTS.
  (~~SOME~~ DOUBLE) (SIMPLE)

- IF f HAS 2 (DISTINCT) FIXED POINTS, THEN f IS CONJUGATE TO A ~~DILATION~~ LINEAR MAP $z \to \alpha z$

  (IF P_1, P_2 ARE THE FIXED POINTS SEND $P_1 \to 0$, $P_2 \to \infty$
  $f'(P_1) = \alpha$, $f'(P_2) = 1/\alpha$)
  (NOTE IN THIS CASE $f'(p) \neq 1$, SINCE
  $(z - f(z))'\big|_{z=p} = 1 - f'(p) \neq 0$)

- IF f HAS A SINGLE (DOUBLE) FIXED POINT, p THEN f IS CONJUGATE TO $z \to z+1$ AND $f'(p) = 1$

  (SEND p TO ∞.
  IN THIS CASE, THE MAP IS
  $z \to az+b$, BUT $a=1$. NOW CONJUGATE BY $w \to \frac{w}{b}$ TO GET $z \mapsto z+1$)



DOUBLE FIXED POINT (PARABOLIC)

TWO (ELLIPTIC) FIXED POINTS $|f'(p)| = 1 = |f'(q)|$

TWO (HYPERBOLIC) FIXED PTS $f'(p) \in \mathbb{R}$, $f'(p) > 1$

(LOXODROMIC) $f'(p) \in \mathbb{C} \setminus \mathbb{R}$, $|f'(p)| \neq 1$

SINCE EACH MÖBIUS MAP HAS AN ASSOCIATED MATRIX $\pm A \in PSL_2(\mathbb{C})$, IT HAS AN INVARIANT

$$\tau(f) = (\operatorname{tr} A)^2$$

THE SQUARE OF THE TRACE OF A (SQUARED TO REMOVE THE ISSUE WITH SIGN)

THM: LET $f \in$ Möb $f \neq id$ HAVE MULTIPLIERS α AND $1/\alpha$ ($\alpha = 1 \Leftrightarrow$ PARABOLIC). THEN

$$\tau(f) = \alpha + \frac{1}{\alpha} + 2$$

FURTHER

- $\tau(f) = 4 \Rightarrow f$ IS PARABOLIC
- $\cancel{0}<\tau(f) < 4 \Rightarrow f$ IS ELLIPTIC
- $\tau(f) > 4 \Rightarrow f$ IS HYPERBOLIC
- $\tau(f) \in \mathbb{C} \setminus [0,\infty) \Rightarrow f$ IS LOXODROMIC

COR NON IDENTITY $f, g \in$ Möb ARE CONJUGATE $\Leftrightarrow \tau(f) = \tau(g)$

Pf/
- IN THE PARABOLIC CASE, f HAS THE ASSOC. MATRIX $\pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ SO $\tau(f) = 4$.
- OTHERWISE, $f(z) = \alpha z$ HAS THE MATRIX $\pm\begin{pmatrix} \sqrt{\alpha} & 0 \\ 0 & 1/\sqrt{\alpha} \end{pmatrix}$

$$\text{SO } \tau(f) = \left(\sqrt{\alpha} + \frac{1}{\sqrt{\alpha}}\right)^2 = \alpha + 2 + 1/\alpha$$