Möbius Functions

Def: A Möbius map (or fractional linear transformation) is a rational function of the form

$$z \mapsto \frac{az+b}{cz+d} \qquad \begin{array}{l} a,b,c,d \in \mathbb{C} \\ ad-bc \neq 0 \end{array}$$

Such maps are biholomorphisms of $\hat{\mathbb{C}}$, with $f^{-1}(z) = \dfrac{dz-b}{-cz+a}$. $f(\infty) = \dfrac{a}{c}$, $f(0) = \dfrac{b}{d}$
(or $f(\infty) = \infty$ if $c=0$) (∞ if $d=0$)

The set of all Möbius maps forms a group under composition (the Möbius group) which can be identified with the matrix group

$$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, ad-bc \neq 0 \right\} . \qquad f_A(z) = \frac{az+b}{cz+d}$$

~~But~~ where $AB \leftrightarrow f_{AB} = f_A \circ f_B$

But, this does not give us a unique identification, since $f_{\alpha A} = f_A$ for any $\alpha \in \mathbb{C}^*$.

So we can normalize matrices to have determinant 1. This almost works, except $\det(\alpha A) = \alpha^2 \det(A)$,

so this is unique up to sign.

SO WE FURTHER IDENTIFY $I_d = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ AND $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

ie $PSL_2(\mathbb{C}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{C},\ ad-bc=1 \right\} / \pm I_d$

$\cong$ Möb.

THERE ARE THREE SPECIAL TYPES OF SUCH MAPS:

- TRANSLATIONS $T_\beta : z \mapsto z+\beta$
- LINEAR MAPS/ DILATIONS $L_\alpha : z \mapsto \alpha z$
- INVERSION $\iota : z \mapsto 1/z$

THM:

(i) MöB IS GENERATED BY THE TRANSLATIONS, LINEAR MAPS, AND INVERSION

(ii) MöB ACTING ON $\hat{\mathbb{C}}$ IS SIMPLY 3-TRANSITIVE

ie, GIVEN ANY $(p_1, p_2, p_3) \in \hat{\mathbb{C}}$, (q_1, q_2, q_3), THERE IS A UNIQUE $\mu \in$ MöB SENDING $p_k \to q_k$ FOR $k = 1, 2, 3$.

(iii) THE ACTION OF MöB PRESERVES CIRCLES IN $\hat{\mathbb{C}}$

~~IN (iii)~~ A CIRCLE IN $\hat{\mathbb{C}}$ IS ANY CIRCLE IN $\mathbb{C}$, ie $\{z \mid |z-p| = r\}$ OR A STRAIGHT LINE IN $\mathbb{C}$ WITH THE POINT ∞ ADDED.

THESE ARE EXACTLY EUCLIDEAN CIRCLES ON $S^2 \subset \mathbb{R}^3$ UNDER STEREOGRAPHIC PROJECTION.

Pf/ (i) GIVEN $f \in \text{Möb}$, $f(z) = \dfrac{az+b}{cz+d}$.

- IF $c = 0$, $f(z) = \dfrac{a}{d} z + \dfrac{b}{d}$ WHICH IS $T_{a/d} \circ L_{b/d}$
- IF $c \neq 0$, ITS JUST UGLIER.

$$f(z) = \frac{a}{c} + \frac{bc-ad}{c^2} \cdot \frac{1}{z + d/c} = T_{a/c} \circ L_{\alpha} \circ (\iota \circ T_{d/c})$$

(ii) (3-TRANSITIVE) IF NO $p_k = \infty$, THEN

$$f(z) = \frac{(p_2 - p_3)(z - p_1)}{(p_2 - p_1)(z - p_3)} \quad \text{SENDS } (p_1, p_2, p_3) \to (0, 1, \infty)$$

(IF $p_k = \infty$, USE $\dfrac{p_2 - p_3}{z - p_3}$, $\dfrac{z - p_1}{z - p_3}$, OR $\dfrac{z - p_1}{p_2 - p_1}$)

FOR UNIQUENESS, LET $(p_1, p_2, p_3) \overset{g}{\longmapsto} (0, 1, \infty)$

WRITE $f \circ g^{-1}(z) = \dfrac{az+b}{cz+d} \Rightarrow b = c = 0,\ a = d$

SO $f \circ g^{-1}(z) = z$ IE $f \equiv g$.

(iii) THAT TRANSLATIONS & LINEAR MAPS PRESERVE CIRCLES IN $\hat{\mathbb{C}}$ IS TRIVIAL. LET'S CHECK FOR ι.

FIRST, CONSIDER A CIRCLE IN $\mathbb{C}$: $|z - p|^2 = r^2 = C_r(p)$. LET $w = 1/z$

THEN $\iota(C_r(p))$ SATISFIES $|1 - aw|^2 = r^2 |w|^2$, IE

$$0 = |1 - aw|^2 - r^2 |w|^2 = (1 - aw)\overline{(1 - aw)} - r^2 |w|^2$$

$$= (|a|^2 - r^2)|w|^2 - aw - \overline{aw} + 1$$

LETTING $w = u + iv$ WITH $u, v \in \mathbb{R}$: $(|a|^2 - r^2)(u^2 + v^2) + Au + Bv + 1 = 0$ WITH $A, B \in \mathbb{R}$.

IF $r = |a|$, THIS IS A STRAIGHT LINE IN $\mathbb{C}$ (A CIRCLE THROUGH ∞).

IF $r \neq |a|$, THE QUADRATIC EQUATION IN u & v IS A CIRCLE IN $\mathbb{C}$. A SIMILAR CALCULATION WORKS FOR LINES IN $\mathbb{C}$.

NOTE THAT $f(z)=\dfrac{az+b}{cz+d} \Rightarrow f'(z) = \dfrac{ad-bc}{(cz+d)^2}$

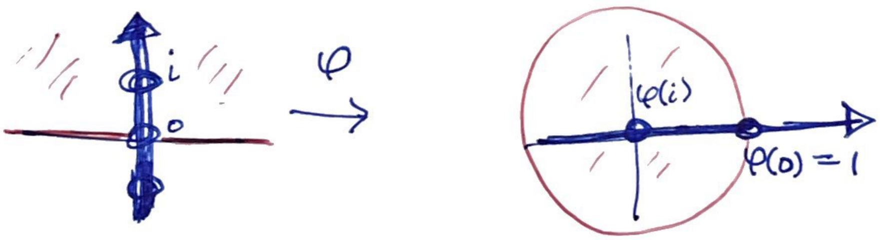
IN OTHER WORDS, $f'(z) \neq 0$ FOR $z \in \mathbb{C}$.

THUS, LOCALLY f PRESERVES ANGLES ~~IE IN A NBHD OF~~ EVERYWHERE, SINCE MÖBIUS MAPS ARE BIHOLOMORPHISMS OF $\hat{\mathbb{C}}$

(IF $p=\infty$ OR $f(p)=\infty$, THEN $f\circ z$ OR $z\circ f$ OR $z\circ f\circ z$ PRESERVES ANGLES AT 0)

THE CAYLEY MAP $z \overset{\varphi}{\longmapsto} \dfrac{i-z}{i+z}$ SATISFIES
$$\varphi(0)=1$$
$$\varphi(i)=0$$
$$\varphi(\infty)=-1$$

IE SENDS THE IMAGINARY AXIS TO $\mathbb{R}$.



THUS φ: UPPER HALF PLANE TO $\mathbb{D}$

$$\mathbb{H} \to \mathbb{D}$$

DEF/ THE CROSS-RATIO OF FOUR DISTINCT POINTS (P_1, P_2, P_3, P_4) IN $\hat{\mathbb{C}}$

IS $$[P_1, P_2, P_3, P_4] = \frac{(P_3-P_1)(P_4-P_2)}{(P_2-P_1)(P_4-P_3)}$$

IF ONE $P_K = \infty$, WE TAKE A LIMIT, IE $[\infty, P_2, P_3, P_4] = \dfrac{P_4-P_2}{P_4-P_3}$, ETC

NOTE $[0,1,P,\infty] = \dfrac{P-0}{1-0} = P$.

IF $P_1 < P_2 < P_3 < P_4$, THEN $[P_1,P_2,P_3,P_4] > 1$.

IF ~~$[P_1,P_2,P_3,P_4] \in \mathbb{R}$, THE FOUR POINTS~~

THM:

(i) IF $f \in$ MÖB MAPS $(p_1, p_2, p_4) \mapsto (0, 1, \infty)$, THEN

$$[p_1, p_2, p_3, p_4] = f(p_3)$$

(ii) $f \in$ MÖB TAKES $(p_1, p_2, p_3, p_4) \to (q_1, q_2, q_3, q_4)$

$$\Longleftrightarrow \quad [p_1, p_2, p_3, p_4] = [q_1, q_2, q_3, q_4]$$

(iii) IF f PRESERVES CROSS-RATIOS OF ALL QUADRUPLES, THEN $f \in$ MÖB

(iv) p_1, p_2, p_3, p_4 LIE ON A CIRCLE IN $\hat{\mathbb{C}} \Longleftrightarrow [p_1, p_2, p_3, p_4] \in \mathbb{R}$.

Pf/(i): THE MAP SENDING (p_1, p_2, p_4) TO $(0,1,\infty)$ $\quad f(z) = \dfrac{(z-p_1)(p_2-p_4)}{(z-p_4)(p_2-p_1)}$

AND CLEARLY GIVES THE CROSS RATIO AS $f(p_3)$.

(ii) LET $\varphi \in$ Möb TAKE $(p_1, p_2, p_4) \mapsto (0, 1, \infty)$

$\psi \in$ Möb TAKE $(q_1, q_2, q_4) \mapsto (0, 1, \infty)$.

IF $\exists\, f \in$ Möb TAKING $(p_1, p_2, p_3, p_4) \to (q_1, q_2, q_3, q_4)$

THEN $\psi \circ f \circ \varphi^{-1}$ IS A MÖBIUS FIXING $0, 1, \infty$,

SO $=$ IDENTITY.

THUS $[p_1, p_2, p_3, p_4] = \varphi(p_3) = \psi(f(p_3)) = \psi(q_3) = [q_1, q_2, q_3, q_4]$

CONVERSELY IF THE CROSS RATIOS AGREE,

$\varphi(p_3) = \psi(q_3)$, SO $f = \psi^{-1} \circ \varphi \in$ Möb IS THE MAP.

(iii) LET $f(P_1)=0$, $f(P_2)=1$, $f(P_4)=\infty$. IF $P_3 \notin \{P_1, P_2, P_4\}$

BY (i) AND (ii)

$$f(z) = [0, 1, f(z), \infty] = [f(P_1), f(P_2), f(z), f(P_4)] = [P_1, P_2, z, P_4]$$

$$= \frac{(z-P_1)(P_4-P_2)}{(P_2-P_1)(P_4-z)},$$ SO f IS MÖBIUS.

(iv) LET f BE THE MÖBIUS MAP SENDING (P_1, P_2, P_4) TO $(0, 1, \infty)$.

SINCE f IS MOBIUS, IT SENDS THE CIRCLE CONTAINING P_1, P_2, P_4 TO $\mathbb{R} \cup \{\infty\}$. THUS, P_3 IS ON THIS CIRCLE $\iff$ $f(P_3) = [P_1, P_2, P_3, P_4] \in \mathbb{R}$.

AUTOMORPHISMS

LET U BE A DOMAIN IN $\hat{\mathbb{C}}$.
$\mathrm{Aut}(U)$ IS THE GROUP OF ALL BIHOLOMORPHISMS $f: U \to U$, WITH THE OPERATION OF COMPOSITION.

$f \in \mathrm{Aut}(U) \leftrightarrow f: U \to U$ IS A HOLOMORPHIC BIJECTION.

THM:

(i) $\mathrm{Aut}(\hat{\mathbb{C}}) = \text{Möb}$

(ii) $\mathrm{Aut}(\mathbb{C}) = \{ f \mid f(z) = az + b,\ a \in \mathbb{C}^*,\ b \in \mathbb{C} \}$.

Pf/(i) WE JUST NEED TO CHECK $f \in \mathrm{Aut}(\mathbb{C}) \Rightarrow f$ IS MÖBIUS.

FIRST, LET g BE THE MÖBIUS MAP WITH $g(f(\infty)) = \infty$

SO $h = g \circ f \in \mathrm{Aut}(\hat{\mathbb{C}})$ WITH $h(\infty) = \infty$.

SINCE h IS INJECTIVE, $h^{-1}(\infty) = \infty$.

BUT h IS A RATIONAL MAP FIXING ∞, AND SO IS A DEGREE 1 POLYNOMIAL. SINCE $f = g^{-1} \circ h$ WITH g, h MÖBIUS, f IS MÖBIUS.