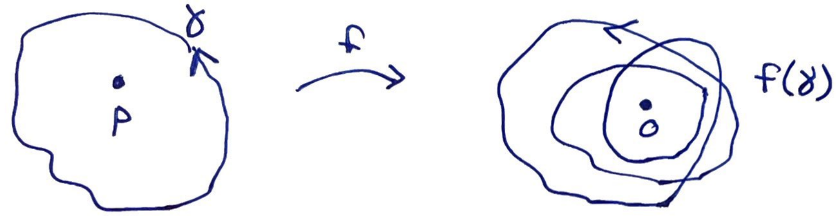
THE PRINCIPLE OF THE ARGUMENT

~~TH~~ AS WE HAVE SEEN, THERE IS A STRONG TIE BETWEEN TOPOLOGICAL AND ANALYTIC DESCRIPTIONS OF HOLOMORPHIC FUNCTIONS.

CONSIDER THE CASE WHERE $f(p) = 0$, AND p IS A ZERO OF ORDER 3. THEN FOR A SUFFICIENTLY SMALL POSITIVELY ORIENTED JORDAN CURVE γ AROUND p,

$z \mapsto f(z)$ BEHAVES LIKE $w \mapsto w^3$ ON $\operatorname{int}(\gamma)$
AND $f(z) = 0$ HAS 3 SOLUTIONS (WITH MULTIPLICITY).



BUT ALSO, WRITING $f(z) = (z-p)^3 f_1(z)$ $\left(f_1(p) \neq 0,\ f_1 \in \mathcal{O}(\mathbb{D}^*_r(p)) \right)$
THE LOGARITHMIC INTEGRAL

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \frac{1}{2\pi i} \int_\gamma \left(\frac{3}{z-p} + \frac{f_1'(z)}{f_1(z)} \right) dz = \frac{1}{2\pi i} \int \frac{3\, dz}{z-p} = 3$$

SINCE $\frac{f_1'}{f_1}$ IS HOLOMORPHIC NEAR p.

THAT IS, LOCALLY, THE

NUMBER OF ROOTS = WINDING NUMBER
= LOGARITHMIC INTEGRAL.

BUT A VERSION OF THIS HOLDS MORE GLOBALLY.

LET $U \subset \mathbb{C}$ BE A DOMAIN WITH $f \in \mathcal{M}(U)$

AND $\gamma : [0,1] \to U$ BE PIECEWISE C', SO THAT

f HAS NO ZEROS OR POLES ON $\{\gamma\}$.

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_0^1 \frac{(f\circ\gamma)'(t)}{(f\circ\gamma)(t)}\,dt = \frac{1}{2\pi i}\int_{f\circ\gamma}\frac{d\xi}{\xi} = W(f\circ\gamma, 0)$$

THIS EXTENDS READILY (BY LINEARITY) TO ANY CHAIN

$$f\circ\gamma = \sum_{k=1}^{m} n_k (f\circ\gamma_k)$$

$$\boxed{\begin{array}{l}\underline{\text{THM:}}\ \text{FOR ANY CHAIN } \gamma \text{ IN A DOMAIN } U \text{ AND} \\ f \text{ MEROMORPHIC ON } U \text{ SO THAT } f \text{ HAS NO ZEROS} \\ \text{OR POLES ON } \{\gamma\}, \\ \qquad \dfrac{1}{2\pi i}\displaystyle\int_\gamma \frac{f'(z)}{f(z)}\,dz = W(f\circ\gamma, 0)\end{array}}$$

THUS, THE LOGARITHMIC INTEGRAL $\frac{1}{2\pi i}\int \frac{f'}{f}$ MEASURES THE NET CHANGE IN THE ARGUMENT OF f AS IT ~~IS~~ TRAVERSES γ.

NOTE IF $\gamma \sim 0$ IN U, ~~THE~~ THE RESIDUE THEOREM CAN BE USED TO COMPUTE $\int_\gamma \frac{f'}{f}$ /

THM (ARGUMENT PRINCIPLE) LET U BE A DOMAIN IN $\mathbb{C}$ WITH f MEROMORPHIC ON U, AND $\gamma \sim 0$ IN U SO THAT f HAS NO ZEROS OR POLES ON $\{\gamma\}$. THEN

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_{p\in f^{-1}(0)} W(\gamma,p)\deg(f,p) - \sum_{p\in f^{-1}(\infty)} W(\gamma,p)\deg(f,p)$$

IN PARTICULAR, LET $D \subset U$ SATISFY $\overline{D}\subset U$. FOR $q\in\hat{\mathbb{C}}$,

LET $$N_f(D,q) = \sum_{p\in f^{-1}(q)\cap D} \deg(f,p)$$

DENOTE THE NUMBER OF SOLUTIONS TO $f(p)=q$ IN D COUNTED WITH MULTIPLICITY.

THEN IF γ IS A POSITIVELY ORIENTED JORDAN CURVE IN U SO THAT $D=\mathrm{int}(\gamma)\subset U$

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = N_f(D,0) - N_f(D,\infty)$$

$$= (\#\text{ZEROS IN } D) - (\#\text{POLES IN } D)$$

PROOF: OBSERVE f IS NOT IDENTICALLY 0 IN U,

(SINCE f IS NONZERO ON $\{\gamma\}$ WITH U CONNECTED)

SO $g(z) = \dfrac{f'(z)}{f(z)}$ IS MEROMORPHIC ON U.

- IF p IS A POLE FOR g, EITHER p IS A ZERO OR A POLE FOR f.

FIRST, LETS CONSIDER THE ZEROS

SPOZE $p \in f^{-1}(0)$, SO $f(z) = (z-p)^m f_1(z)$

$m = \deg(f,p)$, $f_1(p) \neq 0$, $f_1 \in \mathcal{O}(D_r(p))$

SO $$g(z) = \frac{f'(z)}{f(z)} = \frac{m(z-p)^{m-1} f_1(z) + (z-p)^m f_1'(z)}{(z-p)^m f_1(z)} = \frac{m}{z-p} + \frac{f_1'(z)}{f_1(z)}.$$

THE RESIDUE OF g AT p IS m, (SINCE $\frac{f_1'}{f}$ HOLO NEAR p)

$$\operatorname{res}(g,p) = m = \deg(f,p)$$

NOW, THE POLES

IF $q \in f^{-1}(\infty)$, $f(z) = (z-q)^{-m} f_1(z)$, WITH $m = \deg(f,q)$, $f_1(q) \neq 0$.

SINCE $$g'(z) = \frac{-m}{z-q} + \frac{f_1'(z)}{f_1(z)},$$

$$\operatorname{res}(g,p) = -\deg(f,q).$$

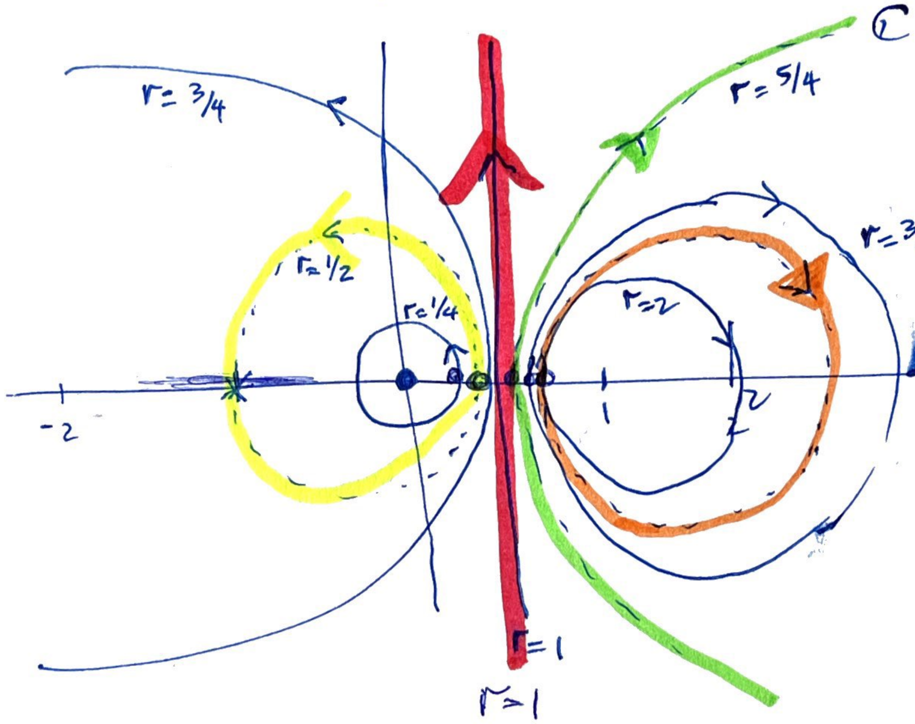
COMBINING THESE WITH THE RESIDUE THM:

$$\frac{1}{2\pi i}\int_\gamma g(z)\,dz = \sum_{p \in f^{-1}(0) \cup f^{-1}(\infty)} W(\gamma,p)\operatorname{res}(g,p)$$

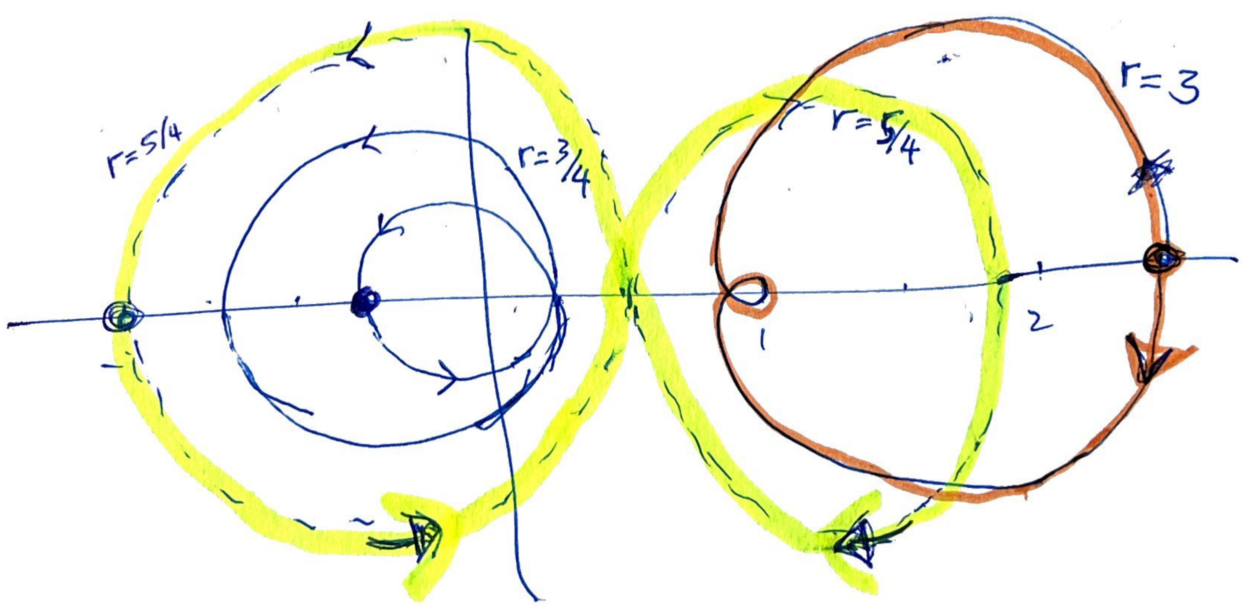
$$= \sum_{p \in f^{-1}(0)} W(\gamma,p)\deg(f,p) - \sum_{q \in f^{-1}(\infty)} W(\gamma,p)\deg(f,q)$$

$\boxtimes$

EXAMPLES CONSIDER THE IMAGE OF $\mathbb{D}_r(0)$ UNDER $z \mapsto \dfrac{z}{z+1}$ AS r INCREASES.



FOR $z \mapsto \dfrac{z^2}{(z+1)(z-2)}$



BY THE ARGUMENT PRINCIPLE, IN THEORY THIS TELLS US HOW TO COUNT THE NUMBER OF ZEROS OF $f \in \mathcal{O}(U)$ ~~WITHOUT KNOWING~~ IN A GIVEN REGION WITHOUT KNOWING THEM EXACTLY.

BUT, IF YOU THINK ABOUT IT, TO COMPUTE

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz$$

, GENERALLY WE'D WANT TO USE RESIDUES, REQUIRING US TO KNOW THE ZEROS ....

THE WAY AROUND THIS IS TO COMPARE $f(z)$ TO A FUNCTION WHOSE ZEROS ARE KNOWN.

ROUCHÉ'S THEOREM LET γ BE A JORDAN CURVE IN A DOMAIN U, WITH $f, g \in \mathcal{O}(U)$ AND $D = \operatorname{int}(\gamma) \subset U$.

THEN, IF

$$|f(z) - g(z)| < |g(z)| \quad \text{FOR } z \in \{\gamma\}$$

f AND g HAVE THE SAME NUMBER OF ZEROS IN D (COUNTING MULTIPLICITIES) IE

$$N_f(D, 0) = N_g(D, 0)$$

Pf/

NEITHER f NOR g CAN VANISH ON $\{\gamma\}$ SINCE THE INEQUALITY IS STRICT. SO THE IMAGES OF γ UNDER f AND g ARE CLOSED CURVES $\eta = f\circ\gamma$ AND $\xi = g\circ\gamma$ IN $\mathbb{C}^*$, WITH $|\eta(t) - \xi(t)| < |\xi(t)|$ FOR ALL t.

IN THIS CASE, THE WINDING NUMBERS ABOUT 0 MUST AGREE FOR η AND ξ:

THEY ARE ~~HOM~~ FREELY HOMOTOPIC IN $\mathbb{C}^*$, SO CONSIDER A HOMOTOPY $H(t,s) = (1-s)\eta(t) + s\,\xi(t)$.

IF $H(t,s) = 0$ FOR SOME $t, s \in [0,1]\times[0,1]$,

THEN $(1-s)\eta(t) + s\xi = 0$ SO

$$(1-s)(\eta(t) - \xi(t)) = (1-s)\eta(t) - \xi(t) + s\,\xi(t) = -\xi(t).$$

THUS $|\eta(t) - \xi(t)| \geq (1-s)|\eta(t) - \xi(t)| = |\xi(t)|$. $\Rightarrow\Leftarrow$

WE MAY ASSUME γ IS POSITIVELY ORIENTED, SO

$$N_f(D,0) = W(\eta, 0) = W(\xi, 0) = N_f(D,0). \quad \boxtimes$$

A MORE GENERAL VERSION OF ROUCHÉ'S THM:

THM LET γ BE A JORDAN CURVE IN U WITH $f, g \in \mathcal{O}(U)$, $D = \text{int}(\gamma)$ AND f, g SATISFY

$$|f(z) - g(z)| < |f(z)| + |g(z)| \quad \text{FOR } z \in \{\gamma\}$$

THEN

$$N_f(D,0) = N_g(D,0)$$

SEE PROBS 36, 37 IN CH3 OF ZAKERI

EXAMPLE: LET $f(z) = z^5 + 3z^2 + 1$.

- FOR $|z| < 2$, f HAS 5 ZEROS: ~~SINCE~~

  TAKE $g(z) = z^5$, $|f-g| = |3z^2+1| \leq 13 < |z^5| = 32$.

  SO $f(z)$ AND z^5 HAVE SAME # OF ZEROS IN $\mathbb{D}_2$, IE $\underline{5}$

- FOR $|z| < 1$, TAKE $g(z) = 3z^2$:

  $$|f-g| = |z^5+1| \leq 2 < |3z^2| = 3$$

  SO TWO ZEROS IN $\mathbb{D}_1$.

USING MAPLE, I GET THE 5 ROOTS TO BE APPROX

$0.773 + 1.19i$, $-0.018 + 0.575i$, -1.51,
$0.773 - 1.19i$, $-0.018 - 0.575i$

ABS: 1.42, 0.575, 1.51