<u>RECALL:</u> IF P IS AN ISOLATED SINGULARITY FOR $f \in \mathcal{O}(U \setminus \{P\})$ AND $\overline{\mathbb{D}_r(P)} \subset U$, THE <u>RESIDUE OF f AT P</u> IS

$$\operatorname{res}(f, P) = \frac{1}{2\pi i} \int_{\mathbb{T}_r(P)} f(z)\,dz \quad \Big(= a_{-1}$$

WHERE $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-P)^n$

<u>THE RESIDUE THEOREM:</u> LET $U \subset \mathbb{C}$ BE OPEN WITH $E \subset U$ HAVING NO ACCUMULATION POINTS IN U, AND $f \in \mathcal{O}(U \setminus E)$, AND LET γ BE A NULL HOMOLOGOUS CYCLE IN $U \setminus E$

THEN
$$\int_\gamma f(z)\,dz = 2\pi i \sum_{P \in E} W(\gamma, P)\operatorname{res}(f, P)$$

IF $E = \emptyset$, THIS IS CAUCHY'S THM.

PROOF: LET $A = \{ p \in E \mid W(\gamma, p) \neq 0 \}$.

A IS BOUNDED, SINCE FOR $p \in (\mathbb{C} \setminus \mathrm{int}(\{\gamma\}))$, $W(\gamma, p) = 0$.

FURTHER A IS FINITE, SINCE OTHERWISE IT WOULD HAVE AN ACCUMULATION POINT $q \in \overline{U}$. SINCE E HAS NO ACCUM. POINTS, $q \in \partial U$. LET D BE A DISK CENTERED AT q WITH $D \cap \{\gamma\} = \emptyset$.

BUT D MEETS $\mathbb{C} \setminus U$, SO $\gamma \sim 0$ IN $U \Rightarrow W(\gamma, \cdot)$ VANISHES IN D

THUS $A \cap D = \emptyset$, AND HENCE q CAN'T BE AN ACCUMULATION POINT OF A. SO $A = \{p_1, \dots, p_n\}$.

HENCE THIS IS A FINITE SUM

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{p \in E} W(\gamma, p)\,\mathrm{res}(f, p) = 2\pi i \sum_{p \in A} W(\gamma, p)\,\mathrm{res}(f, p)$$

CHOOSE r SUFF SMALL THAT $\overline{\mathbb{D}}_r(p_k)$ ARE ALL DISJOINT AND CONTAINED IN $U \setminus \{\gamma\}$ WITH $\overline{\mathbb{D}}_k(p_k) \cap E = \{p_k\}$ FOR EACH k.

THEN $\gamma \sim \eta = \sum_{k=1}^{n} W(\gamma, p_k)\,\Pi_r(p)$ IN $U \setminus E$.

INDEED, IF $p \in \mathbb{C} \setminus U$ OR $p \in E \setminus A$, $\Pi_r(p_k)$ HAS WINDING NUMBER 0 WITH RESP. TO p,

SO $W(\eta, p) = W(\gamma, p)$.

ALSO IF $p_j \in A$, THEN $\Pi_r(p_k)$ HAS $W(\Pi_r(p_k), p) = 0$ IF $j \neq k$, 1 ~~OTHER~~ IF $j = k$.

SO AGAIN $W(\eta, p) = W(\gamma, p)$.

THUS $\int_\gamma f(z)\,dz = \int_\eta f(z)\,dz = \sum_{k=1}^{n} W(\gamma, p_k) \int_{\Pi_r(p_k)} f(z)\,dz = 2\pi i \sum W(\gamma, p_k)\,\mathrm{res}(f, p_k)$

$\boxtimes$

USUALLY, γ IS A JORDAN CURVE!

COR: LET γ BE A POSITIVELY ORIENTED JORDAN CURVE IN A DOMAIN U. SPOZE $\{p_1, \dots, p_k\} \subset \operatorname{int}(\gamma) \subset U$, WITH f HOLOMORPHIC IN $U \setminus \{p_1, \dots, p_k\}$. THEN

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{res}(f, p_k)$$

EX

LET $I = \displaystyle\int_{\Pi_2(0)} \frac{dz}{(z-3)(z^4-1)}$.

THE FUNCTION $f(z)$ HAS SIMPLE POLES AT $\pm i, \pm 1, 3$.

$$\operatorname{res}(f,1) = \lim_{z\to 1} \frac{1}{(z-3)(z+1)(z^2+1)} = \frac{1}{(-2)(2)(2)} = \frac{-1}{8}$$

$$\operatorname{res}(f,-1) = \lim_{z\to -1} \frac{1}{(z-3)(z-1)(z^2+1)} = \frac{1}{(-4)(-2)(2)} = \frac{1}{16}$$

$$\operatorname{res}(f,i) = \lim_{z\to i} \frac{1}{(z-3)(z^2-1)(z+i)} = \frac{1}{(i-3)(-2)(2i)} = \frac{1}{4(1+3i)}$$

$$\operatorname{res}(f,-i) = \lim_{z\to -i} \frac{1}{(z-3)(z^2-1)(z-i)} = \frac{1}{(-i-3)(-2)(-2i)} = \frac{1}{4(1-3i)}$$

SO $I = 2\pi i\left(-\frac{1}{8} + \frac{1}{16} + \frac{1}{4(1+3i)} + \frac{1}{4(1-3i)}\right) = \frac{-\pi i}{40}$

WE CAN ALSO CALCULATE THE RESIDUE AT ∞:

DEF $\operatorname{res}(f,\infty) = \dfrac{-1}{2\pi i}\displaystyle\int_{\Pi_r(0)} f(z)\,dz$ WHERE f IS HOLOMORPHIC ON $|z| > R$ AND $r > R$.

THE MINUS SIGN IS BECAUSE $\Pi_r(0)$ IS NEGATIVELY ORIENTED WRT. ∞

LEMMA: SPOZE f HOLOMORPHIC FOR ALL z WITH $|z|>R$.

i) IF $\sum_{n=-\infty}^{\infty} a_n z^n$ IS THE LAURENT SERIES FOR f WITH $|z|>R$, THEN $\operatorname{res}(f,\infty) = -a_{-1}$

ii) $\operatorname{res}(f,\infty) = \operatorname{res}(g,0)$ WHERE $g(z) = \dfrac{-f(1/z)}{z^2}$

Pf/ (i) FOLLOWS IMMEDIATELY FROM THE DEFINITION.

(ii) OBSERVE THE SERIES FOR $g(z) = \dfrac{-f(1/z)}{z^2} = -\sum_{n=-\infty}^{\infty} a_n z^{-(n+2)}$ IN $\mathbb{D}^*(0,\frac{1}{R})$

SO $\operatorname{res}(g,0) =$ COEFF OF $\frac{1}{z}$ IN SERIES FOR g

$$= -a_{-1} = \operatorname{res}(f,\infty). \qquad \square$$

THM. (SUM OF RESIDUES IS 0 ON $\hat{\mathbb{C}}$). LET $f \in \mathcal{O}(\mathbb{C} - \{p_1,\dots,p_n\})$ WITH $p_j \neq p_k$. THEN

$$\operatorname{res}(f,\infty) + \sum_{k=1}^{n} \operatorname{res}(f,p_k) = 0.$$

Pf/ CHOOSE $r > \max_k |p_k|$. THEN f IS HOLO IN $\{z \in \mathbb{C} \mid |z|>r\}$

$$\text{SO} \quad \operatorname{res}(f,\infty) = -\frac{1}{2\pi i}\int_{\mathbb{T}_r(0)} f(z)\,dz = -\sum_{k=1}^{n} \operatorname{res}(f,p_k) \qquad \square$$

RETURNING TO THE EARLIER EXAMPLE,

$$I = \int_{\mathbb{T}_2(0)} \frac{dz}{(z-3)(z^4-1)} = -2\pi i\left(\operatorname{res}(f,3) + \operatorname{res}(f,\infty)\right)$$

$$\operatorname{res}(f,3) = \lim_{z\to 3}(z-3)f(z) = \lim_{z\to 3}\frac{1}{z^4-1} = \frac{1}{80}$$

SINCE $g(z) = \dfrac{f(1/z)}{z^2} = \dfrac{-z^3}{(1-3z)(1-z^4)}$ HAS A REMOVABLE SING. AT 0, $\operatorname{res}(f,\infty)=0$.

THUS $I = -2\pi i \cdot \left(\frac{1}{80} + 0\right) = \dfrac{-\pi i}{40}$.

SPOZE f FIXES p, THAT IS $f(p)=p$.
p IS A FIXED POINT OF f; EQUIVALENTLY, IT IS A ZERO OF $z-f(z)$.

THE MULTIPLICITY OF p IS THE ORDER OF THE ZERO OF $z-f(z)$, OR EQUIVALENTLY OF THE POLE $\frac{1}{z-f(z)}$. IF p HAS MULTIPLICITY 1, IT IS A SIMPLE FIXED POINT. OBVIOUSLY

$$p \text{ IS SIMPLE} \iff f'(p) \neq 1.$$

DEF: THE INDEX OF f AT AN ISOLATED FIXED POINT $p \in \mathbb{C}$ IS $\mathrm{ind}(f,p) = \mathrm{res}\left(\frac{1}{z-f(z)}, p\right)$.

LET $\sum_1 a_n (z-p)^n$ BE THE SERIES FOR f AT A FIXED PT p, SO $a_0 = p$.

IF $a_1 = f'(p) \neq 1$, p IS A SIMPLE POLE OF $\frac{1}{z-f(z)}$,

$$\text{SO} \quad \mathrm{res}\left(\frac{1}{z-f(z)}, p\right) = \lim_{z\to p} \frac{z-p}{z-f(z)} = \frac{1}{1-a_1} = \frac{1}{1-f'(p)}$$

$$\text{IE} \quad \mathrm{ind}(f,p) = \frac{1}{1-f'(p)} \quad \text{IF } p \text{ IS SIMPLE.}$$

THIS GENERALIZES TO A FIXED POINT AT ∞ (AS IN THE CASE OF POLYNOMIALS) BY CONSIDERING

$$\hat{f}(z) = \frac{1}{f(1/z)} . \quad \text{IF } f(\infty)=\infty, \ \hat{f}(0) = 0$$

AND WE DEFINE $\boxed{\operatorname{ind}(f,\infty) = \operatorname{ind}(\hat{f}, 0)}$

NOTE THAT IF WE SET $f'(\infty) = \hat{f}'(0)$, THEN

$$\frac{1}{1-f'(p)} = \operatorname{ind}(f,p) \quad \text{EVEN IF } p=\infty$$

EX: LET $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_d z^d$ A POLY, $d \geq 2$.

THEN ∞ IS FIXED POINT OF f.

LET $P(z) = z^d f(1/z) = a_0 z^d + \cdots + a_d$, SO $P(0) = a_d$

$$\frac{1}{z-\hat{f}(z)} = \frac{1}{z - z^d/P(z)} = \frac{P(z)}{z(P(z)-z^{d-1})}$$

SIMPLE POLE AT $z=0$,

~~SINCE~~ SO $\operatorname{res}\left(\frac{1}{z-\hat{f}(z)}, 0\right) = \lim_{z\to 0} \frac{z}{z-\hat{f}(z)} = \lim_{z\to 0} \frac{P(z)}{(P(z)-z^{d-1})} = \frac{a_d}{a_d} = 1$

ie $\boxed{\text{FOR A POLY, } \operatorname{ind}(f,\infty) = 1}$

THM (RATIONAL FIXED POINT FORMULA)

LET f BE A NON CONSTANT RATIONAL MAP $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

THEN $\displaystyle\sum_{p \text{ FIXED}}' \operatorname{ind}(f, p) = 1$

COR IF $f : \mathbb{C} \to \mathbb{C}$ IS A POLYNOMIAL OF DEGREE $d \geq 2$, WITH FIXED POINTS $p_1, p_2, \cdots p_d$ THEN

$$\sum_{k=1}^{d}{}' \operatorname{ind}(f, p_k) = 0$$

Pf/ SINCE f IS RATIONAL, IT HAS FINITELY MANY FIXED POINTS (ALTHOUGH THERE COULD BE ZERO, EG $f(z) = z+1$)

LET $E = \{p \mid f(p) = p\}$. $p \in \mathbb{C}$. THEN

$$\sum_{p \in E}' \operatorname{ind}(f, p) = \sum_{p \in E}' \operatorname{res}\left(\frac{1}{z - f(z)}, p\right) = -\operatorname{res}\left(\frac{1}{z - f(z)}, \infty\right)$$

$$= -\operatorname{res}(g, 0)$$

WHERE

$$g(z) = -\frac{1}{z^2} \frac{1}{1/z - f(\frac{1}{z})} = \frac{1}{z(zf(\frac{1}{z}) - 1)}.$$

IF $f(\infty) \neq \infty$, $\lim_{z \to 0} z f(1/z) = 0$, SO g HAS A SIMPLE POLE AT 0 AND $\operatorname{res}(g, 0) = \lim_{z \to 0} z g(z) = -1$.

IF $f(\infty) = \infty$, $\displaystyle\sum_{f(p) = p}' \operatorname{ind}(f, p) = \sum_{p \in E}' \operatorname{ind}(f, p) + \operatorname{ind}(f, \infty)$

$$= -\operatorname{res}(g, 0) + \operatorname{res}(h, 0) = \operatorname{res}(h - g, 0)$$

WITH $h = \frac{1}{z - \tilde{f}(z)}$.

BUT $\displaystyle h(z) - g(z) = \frac{f(1/z)}{zf(1/z) - 1} - \frac{1}{z(zf(1/z) - 1)} = 1/z.$

SO A VERSION OF THIS HOLDS GLOBALLY.