LAURENT SERIES

IF f HAS A POLE OF ORDER m AT $p \in \mathbb{C}$, WE'VE SEEN WE CAN WRITE

$$f(z) = \sum_{n=-m}^{\infty} a_n (z-p)^n \quad \text{IN } \mathbb{D}_r(p).$$

IT SHOULDN'T BE SURPRISING THAT IF p IS AN ESSENTIAL SINGULARITY, WE CAN DO THE SAME WITH INFINITELY MANY NEGATIVE TERMS; OR MORE GENERALLY, IN AN ANNULUS.

THM: (LAURENT, 1843) LET U BE AN OPEN ANNULUS

$$\{ z \in \mathbb{C} \mid R_1 < |z-p| < R_2 \} \quad \text{WITH} \quad 0 \le R_1 < R_2 \le +\infty$$

EVERY $f \in \mathcal{O}(U)$ HAS A POWER SERIES REPRESENTATION

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-p)^n \quad \text{FOR } z \in U$$

WITH

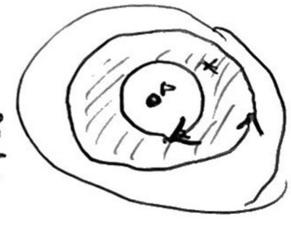
$$a_n = \frac{1}{2\pi i} \int_{\Pi_r(p)} \frac{f(\zeta)}{(\zeta - p)^{n+1}} \, d\zeta \quad \text{FOR ANY } r \in (R_1, R_2)$$

THIS CONVERGES UNIFORMLY IN ANY $\{ z \in \mathbb{C} \mid a < |z-p| < b \}$ WITH $R_1 < a < b < R_2$

Pf/ CHOOSE ANY s, t WITH $R_1 < s < t < R_2$, AND THE CYCLE $\Pi_t(p) - \Pi_s(p)$ IS NULL HOMOLOGOUS ~~$\Pi_s(p)$ AND $\Pi_t(p)$ ARE HOMOLOGOUS~~ WITH WINDING NUMBER 1 AROUND ~~p~~ ANY POINT z WITH $s < |z-p| < t$.

SO WE CAN WRITE

$$g(z) = \frac{1}{2\pi i} \int_{\Pi_t(p)} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad h(z) = \frac{-1}{2\pi i} \int_{\Pi_s(p)} \frac{f(\zeta) d\zeta}{\zeta - z}$$



WITH $f(z) = g(z) + h(z)$, WITH g AND h HOLOMORPHIC OF $\Pi_t(p)$ AND $\Pi_s(p)$ RESP.

SO IN $\mathbb{D}_t(p)$, $\quad g(z) = \sum_{n=0}^{\infty} \alpha_n (z-p)^n$.

ALSO IN $\mathbb{D}_{1/s}(p)$ $\quad \hat{h}(z) = h(p+1/z)$ IS HOLOMORPHIC

WITH $\lim_{z\to 0} \hat{h}(z) = \lim_{z\to\infty} h(z) = 0$,

SO THE SINGULARITY OF $\hat{h}$ AT 0 IS REMOVABLE.

THUS $\hat{h}(z) = \sum_{n=1}^{\infty} \beta_n z^n$ IN $\mathbb{D}_{1/s}(p)$

SO $h(z) = \sum_{n=1}^{\wedge} \beta_n (z-p)^{-n}$ IN $\mathbb{D}_s(p)$ $\left(\text{SINCE } h(z) = \hat{h}\left(\frac{1}{z-p}\right)\right)$

COMBINING THE TWO GIVES US THE BI-INFINITE POWER SERIES FOR f IN THE ANNULUS $\{z \mid s < |z-p| < t\}$.

THIS CONVERGES UNIFORMLY IN THE ANNULUS, SINCE THE SERIES FOR g CONV. UNIF IN $\mathbb{D}_t(p)$ AND THE ONE FOR h CONV. UNIF. FOR $|z-p| > s$.

FINALLY, TO VERIFY THE FORMULAE FOR a_n, CHOOSE ANY RADIUS $r \in (s,t)$ AND SINCE THE SERIES IS UNIF. CONVERGENT ON $\Pi_r(p)$ WE CAN INTEGRATE TERM-BY-TERM:

$$\int_{\Pi_r(p)} \frac{f(\zeta)}{(\zeta-p)^{n+1}}\,d\zeta = \sum_{k=-\infty}^{\infty} a_n \left(\int_{\Pi_r(p)} (\zeta-p)^{k-n-1}\,d\zeta \right)$$

BUT THE INTEGRAL VANISHES UNLESS $k=n$,

AND WHEN $k=n$, WE GET $2\pi i$.

THIS DOESN'T DEPEND ON THE CHOICE OF r SINCE THE CIRCLES ARE ALL HOMOLOGOUS.

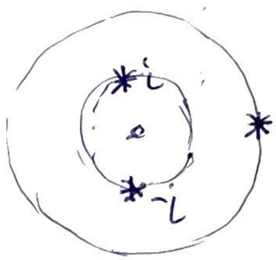
IN GENERAL, IT IS EASIER TO JUST USE KNOWN SERIES TO BUILD LAURENT SERIES, RATHER THAN COMPUTING THE a_n VIA INTEGRATION.

EG:
- $e^{1/z}$ IS HOLO. IN $\mathbb{C}^*$ WITH AN ESSENTIAL SINGULARITY AT 0.
  USING THE SERIES FOR e^z, WE EASILY GET

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots \qquad z \in \mathbb{C}^*$$

- FOR SEVERAL NON-CONCENTRIC SINGULARITES, WE REPRESENT OUR FUNCTION ON DISKS AND ANNULI CENTERED AT SOME POINT $p \in \mathbb{C}$ AND ∞.

FOR EXAMPLE, $f(z) = \dfrac{2z^2+z}{(1+z^2)(z-2)}$ HAS SINGULARITIES AT $\pm i$, 2.



SO WE HAVE LAURENT SERIES ABOUT $z=0$ FOR $|z|<1$, $1<|z|<2$, $|z|>2$.

USING PARTIAL FRACTIONS, $f(z) = \dfrac{1}{1+z^2} + \dfrac{2}{z-2}$.

(*) IF $|z|<1$, $\dfrac{1}{1+z^2} = \dfrac{1}{1-(-z^2)} = \sum_{n=0}^{\infty} (-1)^n z^{2n} = 1 - z^2 + z^4 - z^6 + \cdots$

(**) FOR $|z|>1$, $\dfrac{1}{1+z^2} = \dfrac{1}{z^2}\left(\dfrac{1}{1-(-1/z^2)}\right) = \dfrac{1}{z^2}\sum_{n=0}^{\infty} \dfrac{(-1)^n}{z^{2n}} = \dfrac{1}{z^2} - \dfrac{1}{z^4} + \dfrac{1}{z^6} - \cdots$

(☾) FOR $|z|<2$, $\dfrac{2}{z-2} = -\left(\dfrac{1}{1-z/2}\right) = -\sum_{n=0}^{\infty} \dfrac{z^n}{2^n} = -1 - \dfrac{z}{2} - \dfrac{z^2}{4} - \dfrac{z^3}{8} - \dfrac{z^4}{16} - \cdots$

(☾☾) FOR $|z|>2$ $\dfrac{2}{z-2} = \dfrac{2}{z}\left(\dfrac{1}{1-2/z}\right) = \dfrac{2}{z}\sum_{n=0}^{\infty} \dfrac{2^n}{z^n} = \dfrac{2}{z} + \dfrac{4}{z^2} + \dfrac{8}{z^3} + \dfrac{16}{z^4} + \cdots$

SO FOR $|z|<1$, WE ADD (*) AND (☾):

$$f(z) = \sum_{n=0}^{\infty} (-1)^n z^{2n} + \left(-\sum_{n=0}^{\infty} \frac{z^n}{2^n}\right) = -\frac{z}{2} - \frac{5}{4}z^2 - \frac{z^3}{8} + \frac{15}{16}z^4 - \frac{z^5}{32} - \cdots$$

FOR $1<|z|<2$, ADD (**) AND (☾):

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{2n+2}} - \sum_{n=0}^{\infty} \frac{z^n}{2^n} = \cdots + \frac{1}{z^6} - \frac{1}{z^4} + \frac{1}{z^2} - 1 - \frac{z}{2} - \frac{z^3}{8} - \frac{z^4}{16} - \cdots$$

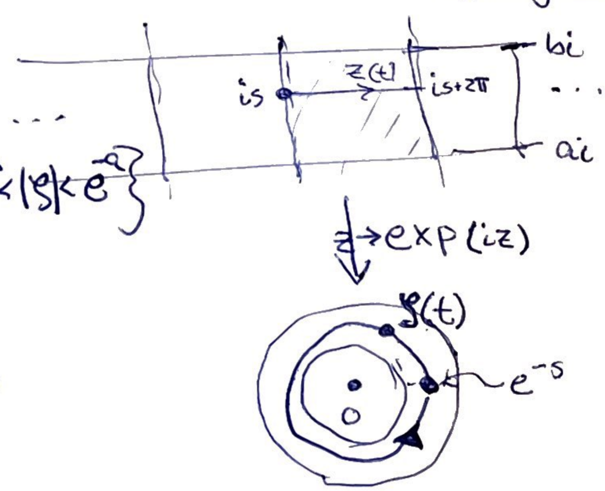
FOR $|z|>2$, ADD (**) AND (☾☾):

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{2n+2}} + \sum_{n=0}^{\infty} \frac{2^{n+1}}{z^{n+1}} = \frac{2}{z} + \frac{5}{z^2} + \frac{8}{z^3} + \frac{15}{z^4} + \frac{32}{z^5} + \cdots$$

WE CAN USE LAURENT SERIES TO OBTAIN FOURIER SERIES:

SPOZE f IS 2π PERIODIC, IE $f(z) = f(z+2\pi)$ FOR $z \in \underset{\text{STRIP}}{S} = \{z \in \mathbb{C}, a<\operatorname{Im} z<b\}$

AND $f \in \mathcal{O}(S)$



NOW LET $\zeta = \varphi(z) = e^{iz}$

$\varphi(S)$ IS AN ANNULUS $U = \{\zeta \mid e^{-b} < |\zeta| < e^{-a}\}$

AND THERE IS $g \in \mathcal{O}(U)$ SO THAT $f = g \circ \varphi$ IN ALL OF S

(DEFINE ON CENTRAL BLOCK, SPREAD BY PERIODICITY OF f)

NOW, IF $g(\zeta) = \sum_{n=-\infty}^{\infty} a_n \zeta^n$ IS THE LAURENT SERIES IN U FOR g,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n e^{inz}$$ GIVES THE FOURIER SERIES FOR f IN STRIP S

a_n ARE FOURIER COEFFS ($\hat{f}(n)$ NOT UNCOMMON)

FOR $a < s < b$, LET $r = e^{-s}$, SO $e^{-b} < r < e^{-a}$

$$a_n = \hat{f}(n) = \frac{1}{2\pi i} \int_{\mathbb{T}_r} \frac{g(\zeta)}{\zeta^{n+1}}\, d\zeta$$

IF WE PARAMETERIZE $z(t)$ BY $z(t) = t + is$, THEN $[is, is+2\pi] \in S$ AND $\varphi(z(t)) = re^{it} = \zeta(t)$ PARAMETERIZES $\mathbb{T}_r$ IN THE STANDARD WAY.

SO

$$\hat{f}(n) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{g(\zeta(t))}{(\zeta(t))^{n+1}} \zeta'(t)\, d\zeta = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(z(t))}{e^{i(n+1)z(t)}} e^{iz(t)} z'(t)\, dt$$

$$= \frac{1}{2\pi} \int_{[is, is+2\pi]} f(z) e^{-inz}\, dz.$$

THIS EASILY GENERALIZES TO ω-PERIODIC FUNCTIONS.

THM LET $\omega \in \mathbb{C}^*$ SPOZE $f(z) = f(z+\omega)$, AND S IS A STRIP IN $\mathbb{C}$ WHICH IS INVARIANT UNDER $z \mapsto z + \omega$, WITH $F \in \mathcal{O}(S)$.

THEN F HAS A FOURIER REPRESENTATION

$$f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n z/\omega} \quad \text{FOR } z \in S.$$

THIS CONVERGES UNIFORMLY IN ANY CLOSED SUBSTRIP OF S

THE FOURIER COEFFS ARE GIVEN BY

$$\hat{f}(n) = \frac{1}{\omega} \int_{[p, p+\omega]} f(z) e^{-2\pi i n z/\omega}\, dz$$

FOR ANY $p \in S$

FOLLOWS FROM EARIER, CONSIDERING $f\left(p + \frac{\omega z}{2\pi}\right)$

RESIDUES

DEF: LET p BE AN ISOLATED SINGULARITY OF f, WITH f HOLOMORPHIC IN AN (OPEN) DELETED NBHD U OF p. THE <u>RESIDUE</u> OF f AT p IS

$$\operatorname{res}(f,p) = \frac{1}{2\pi i}\int_{\mathbb{T}_r(p)} f(z)\,dz$$

WHERE $r>0$ IS ANY RADIUS SO THAT $\overline{\mathbb{D}}_r(p) \subset U$

OF COURSE WE CAN REPLACE THE CIRCLE OF RADIUS r ABOVE WITH ANY POSITIVELY ORIENTED JORDAN CURVE γ IN U WITH $p \in \operatorname{int}(\gamma)$

THM: LET $f \in \mathcal{O}(\mathbb{D}_r^*(p))$ HAS THE LAURENT SERIES

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-p)^n .$$ THEN $\operatorname{res}(f,p) = a_{-1}$

SPOZE f HAS A POLE OF ORDER m AT p, SO THAT

$f_1(z) = (z-p)^m f(z)$ HAS A REMOVABLE SINGULARITY AT p, $f_1(p) \neq 0$.

THEN $f_1(z) = \sum_{n=0}^{\infty} b_n (z-p)^n$ SO $f(z) = \sum_{n=0}^{\infty} b_n (z-p)^{n-m}$ IS THE LAURENT SERIES FOR f, AND $\operatorname{res}(f,p) = b_{m-1} = \dfrac{f_1^{(m-1)}(p)}{(m-1)!}$

~~OR~~ IE

$$\operatorname{res}(f,p) = \frac{1}{(m-1)!}\lim_{z\to p}\left[(z-p)^m f(z)\right]^{(m-1)}$$ ← DERIV, NOT POWER.

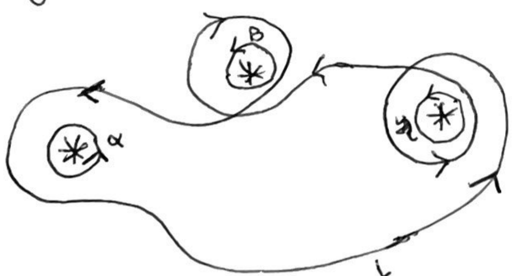
EX: $f(z) = \dfrac{1}{z^2(z+1)}$ WITH A POLE OF ORDER 2 AT $z=0$
SIMPLE AT $z=-1$

$$\operatorname{res}(f,-1) = \lim_{z\to -1}\left((z+1)f(z)\right) = \frac{1}{(-1)^2} = 1.$$

$$\operatorname{res}(f,0) = \lim_{z\to 0}\left[z^2 f(z)\right]' = \lim_{z\to 0} \frac{-1}{(z+1)^2} = -1$$

ALTERNATIVELY, WE CAN USE RESIDUES TO COMPUTE INTEGRALS:

SPOZE f, γ AS IN FIGURE, f HAS SINGULARITIES AT $*$ $p_\alpha, p_\beta, p_\gamma$
γ AS SHOWN.



LET α, β, η BE CURVES AROUND $*$, SO

$$\gamma \sim \alpha - \beta + 2\eta$$

THEN $$\int_\gamma f(z)\,dz = \int_\alpha f(z)\,dz - \int_\beta f(z)\,dz + 2\int_\eta f(z)\,dz$$

$$= 2\pi i\left(\operatorname{res}(f,p_\alpha) - \operatorname{res}(f,p_\beta) + 2\operatorname{res}(f,p_\eta)\right)$$

MORE GENERALLY,

THM: (THE RESIDUE THM): LET $U \subset \mathbb{C}$ OPEN, WITH $E \subset U$ HAVING NO ACCUMULATION POINTS IN U, $f \in \mathcal{O}(U \setminus E)$.
LET $\gamma \sim 0$ WITH $\{\gamma\} \subset U \setminus E$ BE A CYCLE. THEN

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{p\in E} W(\gamma,p)\operatorname{res}(f,p)$$

NOTE $E = \emptyset$ IS CAUCHY'S THM.