LAST TIME, STARTED DISCUSSING (ISOLATED) SINGULARITIES.

- REMOVABLE, eg $\frac{\sin z}{z}$ NEAR 0
- NOT REMOVABLE (A POLE) eg $1/z^2$ NEAR 0

  OR $e^{1/z}$ AT ZERO (SEE BELOW).

~~LET'S EXAMINE A R~~

IF $\lim_{z \to p} f(z)$ EXISTS (NOT ∞) AND f HOLO IN $\mathbb{D}_r^*(p)$, THEN BY MORERA'S THM, p IS A REMOVABLE SINGULARITY. BUT LESS IS ENOUGH:

THM: RIEMANN (1851)

IF A HOLOMORPHIC FUNCTION IS BOUNDED IN A NEIGHBORHOOD OF AN ISOLATED SINGULARITY, THE SINGULARITY IS REMOVABLE

Pf/ SPOSE $f \in \mathcal{O}(\mathbb{D}_r^*(p))$ AND f BOUNDED THERE.

THEN $g(z) = (z-p)^2 f(z) \in \mathcal{O}(\mathbb{D}_r^*(p))$ WITH $\lim_{z \to p} g(z) = 0$.

SO g CAN BE EXTENDED TO $\mathbb{D}_r(p)$ BY SETTING $g(p) = 0$.

SINCE f IS BOUNDED, $\lim_{z \to p} \frac{g(z) - g(p)}{z-p} = \lim_{z \to p} (z-p) f(z) = 0$,

SO $g'(p) = 0$, AND IN $\mathbb{D}_r(p)$, $g(z) = \sum_{n=2}^{\infty} a_n (z-p)^n$.

THUS CAN EXTEND f AS $\sum_{n=0}^{\infty} a_{n+2} (z-p)^n$ ON $\mathbb{D}_r(p)$. $\blacksquare$

NOTE/ PROOF STILL WORKS IF JUST $\lim_{z \to p} (z-p) f(z) = 0$.

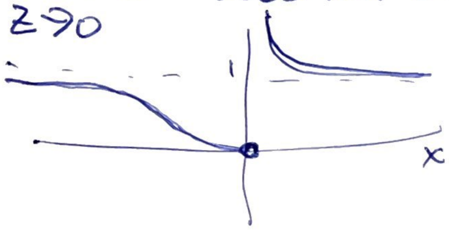
EX: SPOSE $|f(z)| < C|z|^{\alpha}$, SOME CONSTANT $C \in \mathbb{C}$, $-1 < \alpha < 0$.

NOT BDD NEAR 0, PROOF STILL WORKS.

~~EX~~ANOTHER EXAMPLE

CONSIDER $e^{1/z}$ IN A NEIGHBORHOOD OF THE ORIGIN.

$\lim\limits_{z \to 0} e^{1/z}$ DOES NOT EXIST, SINCE FOR $x \in \mathbb{R}$



$$\lim_{x \to 0^+} e^{1/x} = +\infty \qquad \lim_{x \to 0^-} e^{1/x} = 0$$

BUT NOTICE THAT FOR ANY $w \in \mathbb{C}$, THERE IS A SEQUENCE $z_n \to 0$ WITH $\exp(1/z_n) \to w$:

- IF $w = 0$, TAKE $z_n = -1/n$.
- IF $w \neq 0$, FIND $z \neq 0$ WITH $\exp(1/z) = w$ AND LET $z_n = \dfrac{1}{1 + 2\pi i n z}$.

$e^{1/z}$ HAS AN ESSENTIAL SINGULARITY AT $z = 0$.

THM: (CLASSIFICATION OF ISOLATED SINGULARITIES) —

LET p BE AN ISOLATED SINGULARITY OF A HOLOMORPHIC FUNCTION f. THEN p IS ONE OF

(i) A REMOVABLE SINGULARITY FOR WHICH $\lim\limits_{z \to p} f(z)$ EXISTS IN $\mathbb{C}$.

(ii) A POLE WITH $\lim\limits_{z \to p} f(z) = \infty$. IN THIS CASE, FOR SOME $m \in \mathbb{Z}^+$, $f_1(z) = (z-p)^m f(z)$ HAS A REMOVABLE SINGULARITY AT p, WITH $f_1(p) \neq 0$.

(iii) AN ESSENTIAL SINGULARITY WITH $\lim\limits_{z \to p} f(z)$ DOES NOT EXIST (AND IS NOT ∞)

IN THIS CASE, FOR EVERY SMALL $r > 0$, $f(D_r^*(p))$ IS DENSE IN $\mathbb{C}$.

PART (iii) IS THE <u>CASORATI-WEIRSTRASS THM</u> (1868-1876)

BUT IT HAS A STRONGER VERSION:

<u>PICARD'S $\left\{\begin{matrix}\text{BIG}\\ \text{GREAT}\end{matrix}\right\}$ THM (1880)</u>

> LET f BE HOLOMORPHIC IN $\mathbb{D}_r^*(p)$ WITH p AN ESSENTIAL SINGULARITY. THEN WITH AT MOST ONE EXCEPTION, f TAKES ON EVERY VALUE IN $\mathbb{C}$ INFINITELY OFTEN

(WE WON'T PROVE THIS; IT REQUIRES THE THEORY OF NORMAL FAMILIES)

<u>PICARD'S LITTLE THM</u>

> A NON-CONSTANT ENTIRE FUNCTION ASSUMES EVERY VALUE IN $\mathbb{C}$ WITH AT MOST ONE EXCEPTION

<u>PROOF OF THM:</u> SPOZE (iii) FAILS. THEN THERE ARE DISKS $\mathbb{D}_r(p)$ AND $\mathbb{D}_r(q)$ WITH $f(\mathbb{D}_r^*(p))$ DISJOINT FROM $\mathbb{D}_r(q)$. THUS $g(z) = \dfrac{1}{f(z)-q}$ HOLOMORPHIC IN $\mathbb{D}_r^*(p)$ AND BOUNDED BY $1/s$, SOME s.

THUS p IS REMOVABLE FOR g, IE $g \in \mathcal{O}(\mathbb{D}_r(p))$

IF $g(p) \neq 0$, f IS BOUNDED AND SO p IS REMOVABLE FOR f $\Rightarrow$ (i) HOLDS

IF $g(p)=0$ BUT NOT IDENTICALLY 0, WRITE $g(z) = (z-p)^m g_1(z)$ WITH $g_1(p) \neq 0$. (IN FACT $g_1(z) \neq 0$ IN $\mathbb{D}_r(p)$)

THUS $\frac{1}{g_1(z)} \in \mathcal{O}(\mathbb{D}_r(p))$

LET $f_1(z) = (z-p)^m f(z)$

$$f_1(z) = g(z-p)^m + \frac{(z-p)^m}{g(z)} =$$

$$= g(z-p)^m + \frac{1}{g_1(z)}$$

WITH A REMOVABLE SING. AT P,

$f_1(p) = \frac{1}{g_1(p)} \neq 0$, SO CASE (ii). [A POLE]

THE INTEGER m IS UNIQUELY DEFINED. $\blacksquare$

DEF: LET f HAVE A POLE AT p.

THE INTEGER m ABOVE IS THE ORDER OF THE POLE.

IF $\operatorname{ord}(f,p)=1$, p IS A SIMPLE POLE.

IN $\mathbb{D}_r(p)$, WE HAVE $(z-p)^m f(z) = \sum_{n=0}^{\infty} b_n (z-p)^n$,

WITH $b_0 \neq 0$.

SO IN $\mathbb{D}^*_r(p)$, $f(z) = \sum_{n=0}^{\infty} b_n (z-p)^{n-m}$.

THE RATIONAL FUNCTION

$$\frac{b_0}{(z-p)^m} + \frac{b_1}{(z-p)^{m-1}} + \cdots + \frac{b_{m-1}}{(z-p)}$$

IS THE PRINCIPAL PART OF f AT p

THIS IS A POLYNOMIAL P IN $\frac{1}{z-p}$ WITH $P(0)=0$ AND $f(z) - P(\frac{1}{z-p})$ HAS A REMOVABLE SING. AT p

EX: ~~[illegible]~~ IN $\mathbb{C}^*$,

$$\frac{e^z}{z^3} = \underbrace{\frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2z}}_{\text{THE PRINCIPLE PART OF } e^z/z^3 \text{ AT } z=0.} + \frac{1}{3!} + \frac{z}{4!} + \frac{z^2}{5!} + \cdots$$

FUNCTIONS WITH ISOLATED POLES ARE MEROMORPHIC

$$\boxed{\begin{array}{l}\textbf{DEF:} \text{ LET } U \subset \mathbb{C} \text{ BE OPEN, AND } E \subset U \text{ HAVE NO} \\ \text{ACCUMULATION POINTS } (E = \emptyset \text{ IS OK}) \\ \text{IF } f \in \mathcal{O}(U \setminus E) \text{ AND } f \text{ HAS A POLE AT EACH } z \in E, \\ \text{THEN } f \text{ IS } \underline{\text{MEROMORPHIC ON } U} \\ f \in \mathcal{M}(U) \qquad\qquad (\text{SO } \mathcal{O}(U) \subset \mathcal{M}(U))\end{array}}$$

$$\boxed{\begin{array}{l}\underline{\text{FACT}}\text{: EVERY MEROMORPHIC FUNCTION } f \in \mathcal{M}(U) \\ \text{CAN BE WRITTEN AS THE RATIO OF TWO HOLOMORPHIC} \\ \text{FUNCTIONS } p, q \in \mathcal{O}(U).\end{array}}$$

PROOF REQUIRES MORE TOOLS THAN WE HAVE YET

IF U IS A DOMAIN $\mathcal{M}(U)$ IS A FIELD (USING POINTWISE ADDITION, MULTIPLICATION, DIVISION). IT IS THE "FIELD OF FRACTIONS" OF THE RING $\mathcal{O}(U)$.

THE RIEMANN SPHERE

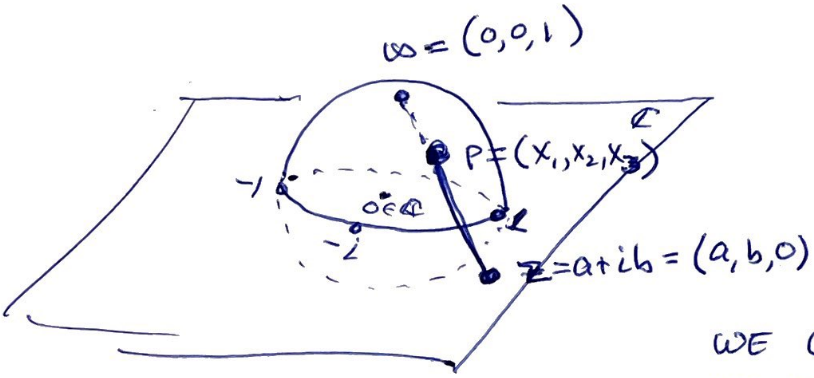
IT IS CONVENIENT TO COMPACTIFY $\mathbb{C}$ BY ADDING A POINT $\{\infty\}$. THIS GIVES THE RIEMANN SPHERE

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \qquad (\text{ALSO } \mathbb{P}^1, \overline{\mathbb{C}})$$

DEF $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ IS THE RIEMANN SPHERE, WITH THE TOPOLOGY GENERATED BY OPEN DISKS $\mathbb{D}_r(p)$ FOR $p \in \mathbb{C}$, $r > 0$ AND $\{z \in \mathbb{C} \mid |z| > r\} \cup \{\infty\}$, $r > 0$

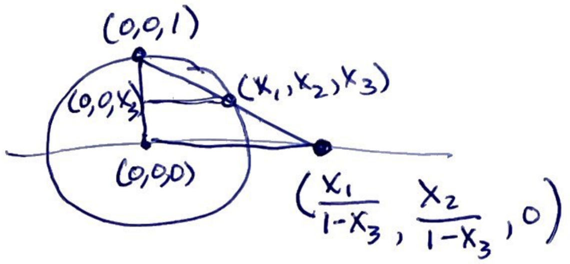
WE CAN EXPLICITLY SEE A HOMEOMORPHISM $\hat{\mathbb{C}} \leftrightarrow S^2$ AS FOLLOWS. WRITE S^2 AS $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$

AND VIEW $\mathbb{C} \Leftrightarrow \{(a, b, 0) \in \mathbb{R}^3\}$ WITH $z = a + ib$



CONNECT EACH POINT IN THE PLANE $\mathbb{C}$ WITH $\infty \in S^2$ WHERE $\infty = (0,0,1)$.
THIS LINE INTERSECTS S^2 AT A POINT $P = (x_1, x_2, x_3)$

GIVEN P, WE CAN CALCULATE (a, b) USING SIMILAR TRIANGLES



IF $x_3 \neq 1$, $\quad z = \dfrac{x_1 + i x_2}{1 - x_3}$

IF $x_3 = 1$, $\quad z = \infty$

AND THEN THIS IS INVERTIBLE

$$\Phi^{-1}(z) = \begin{cases} \left(\dfrac{z+\bar{z}}{1+|z|^2}, \dfrac{z-\bar{z}}{1+|z|^2}, \dfrac{|z|^2-1}{1+|z^2|} \right) & z \neq \infty \\ (0,0,1) & z = \infty \end{cases}$$

ACTUALLY, THIS MAP IS ORIENTATION REVERSING.

THIS CAN BE FIXED BY $(x_1, x_2, x_3) \leftrightarrow \dfrac{x_1 - i x_2}{1 - x_3}$.

WE CAN EXTEND ~~OUR~~ OUR NOTION OF HOLOMORPHIC TO $U \subset \hat{\mathbb{C}}$ OPEN: $f : U \to \hat{\mathbb{C}}$ IS HOLOMORPHIC IF FOR EVERY $p \in U$, $\exists\ \mathbb{D}_r(p)$ IN WHICH f IS HOLOMORPHIC, $f(p) = q$.

IF $p, q \in \mathbb{C}$ NOTHING HAS CHANGED.

OTHERWISE WE COMPOSE f WITH ~~[illegible]~~ $\frac{1}{z}$ TO BRING INFINITY TO 0.

- WHEN $p \neq \infty$, $q = \infty$, IF $\dfrac{1}{f(z)}$ IS HOLO IN A NBHD OF p.
- WHEN $p = \infty$, $q \neq \infty$, IF $f(\frac{1}{z})$ IS HOLO IN A NBHD OF 0
- WHEN $p = \infty$, $q = \infty$, IF $\dfrac{1}{f(1/z)}$ IS HOLO IN A NBHD OF 0

THUS, WE CAN VIEW MEROMORPHIC MAPS AS HOLOMORPHIC MAPS $f : U \to \hat{\mathbb{C}}$.

IF $f(z) = \frac{p(z)}{q(z)}$ WHERE p, q POLYNOMIALS HAVE NO COMMON FACTORS AND $q(z)$ NOT IDENTICALLY ZERO,

THE RATIONAL FUNCTION EXTENDS TO A HOLOMORPHIC MAP $f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ WHERE WE SET $f(\rho) = \infty$ WHERE $q(\rho) = 0$, AND $f(\infty) = \lim_{z \to \infty} f(z)$.

SUCH RATIONAL FUNCTIONS ARE THE ONLY NONTRIVIAL HOLOMORPHIC MAPS FROM $\hat{\mathbb{C}}$ TO $\hat{\mathbb{C}}$:

THM: IF $f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ IS HOLOMORPHIC AND NOT IDENTICALLY ∞, f IS A RATIONAL FUNCTION

Pf/ NOTE $f^{-1}(\infty)$ MUST BE A FINITE SET (OR ELSE IT WOULD HAVE AN ACCUMULATION POINT IN $\hat{\mathbb{C}}$ AND HENCE $f(z) \equiv \infty$)

SO LET $f^{-1}(\infty) = \{p_1, p_2, \cdots, p_n\}$.

REPLACE f BY $1/f$ IF NECESSARY, WE CAN ASSUME $f(\infty) \neq \infty$, SO $p_k \neq \infty$ FOR ALL $k \in \{1, 2, \cdots, n\}$.

RESTRICT f TO $\mathbb{C}$, AND THIS IS MEROMORPHIC WITH POLES AT p_k. LET $P_k\left(\frac{1}{z - p_k}\right)$ BE THE PRINCIPAL PART OF f AT EACH p_k, WITH $P_k(0) = 0$ AND $\deg P_k = \mathrm{ord}(f, p_k)$

LET $g(z) = f(z) - \sum_{k=1}^{n} P_k\left(\frac{1}{z-p_k}\right)$

g HAS REMOVABLE SINGULARITIES AT EACH p_k, SO g EXTENDS TO AN ENTIRE FUNCTION.

ALSO $\lim_{z\to\infty} g(z) = \lim_{z\to\infty} f(z) - \sum_{k=1}^{n} P_k(0) = f(\infty) \neq \infty$

SO $g(z)$ IS BOUNDED FOR $|z| > R$ SOME R.

HENCE, BY LIOUVILLE'S THM, $g(z)$ IS A CONSTANT.

THIS MEANS $f(z)$ IS THE SUM OF ITS PRINCIPAL PARTS AT POLES p_k PLUS A CONSTANT, AND IS HENCE RATIONAL.