<u>HOMOLOGY</u> VS HOMOTOPY.

ONE ISSUE WITH HOMOTOPY IS THAT FIXING THE BASEPOINT INDUCES A LACK OF COMMUTIVITY. IE

$\eta \cdot \gamma \neq \gamma \cdot \eta$ IN ALL CASES, BUT

$$\int_{\eta \cdot \gamma} f(z)\,dz = \int_{\eta} f(z)\,dz + \int_{\gamma} f(z)\,dz = \int_{\gamma \cdot \eta} f(z)\,dz.$$

AS LONG AS $f : U \to \mathbb{C}$ IS CONTINUOUS, EVEN IF $[\gamma]$ AND $[\eta]$ DON'T COMMUTE IN $\pi_1(U, p)$.

WE CAN ABELENIZE HOMOTOPY ~~GROUP~~ TO OBTAIN HOMOLOGY

GIVEN ANY SET S WE CAN FORM THE <u>FREE ABELIAN GROUP GENERATED BY S</u> $\mathcal{G}(S)$

~~LET ... IF $y \neq x$ ... $y = x$. AND THEN~~

NOTE THAT FUNCTIONS $\varphi : S \to \mathbb{Z}$ WITH $\varphi(x) = 0$ FOR ALL BUT FINITELY MANY $x \in S$ FORM AN ABELIAN GROUP, GENERATED BY ELEMENTS OF THE FORM φ_x WHICH TAKES ON THE VALUE 1 AT x AND 0 ELSEWHERE

WRITING x_i FOR φ_{x_i}, WE HAVE A <u>UNIQUE</u> REPRESENTATION OF ANY $\varphi \in \mathcal{G}(S)$ AS

$$\varphi = n_1 x_1 + n_2 x_2 + \cdots + n_k x_k \quad \text{WITH } n_i \in \mathbb{Z} \setminus \{0\},\ x_k \in S.$$

(EXCEPT FOR $0 \in \mathcal{G}(S)$ WHICH IS THE EMPTY SUM)

IF $\gamma = \sum n_i x_i$ IS A CHAIN, $-\gamma$ ~~IS~~ $\sum (-n_i) x_i$ IS ITS INVERSE

DEF

LET $U \subset \mathbb{C}$ BE NONEMPTY AND OPEN.

A <u>CHAIN</u> (OF CURVES) <u>IN U</u> IS A FORMAL SUM

$$\gamma = n_1\gamma_1 + n_2\gamma_2 + \cdots + n_m\gamma_m$$

WITH $\gamma_k : [0,1] \to U$ A CURVE, $n_k \in \mathbb{Z}$ IS ITS <u>MULTIPLICITY</u>

THE IMAGE OF γ IS $\{\gamma\} = \{\gamma_1\} \cup \{\gamma_2\} \cup \cdots \cup \{\gamma_m\}$.

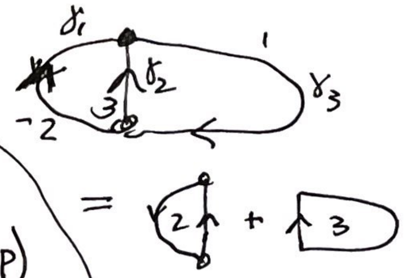
A CHAIN IS A <u>CYCLE</u> IF $\displaystyle\sum_{\gamma_k(0)=p} n_k = \sum_{\gamma_k(1)=p} n_k$

IE ANY INITIAL POINT p OCCURS THE SAME NUMBER OF TIMES AS AN ENDPOINT

MENTION BOUNDARY MAP $\partial\left(\sum n_k\gamma_k\right) = \sum_k n_k\gamma_k(1) - \sum_k n_k\gamma_k(0)$

SO γ IS A CYCLE $\iff \partial\gamma = 0$

EVERY CYCLE CAN BE DECOMPOSED INTO SUMS OF CLOSED CURVES.



IF $\displaystyle\gamma = \sum_{k=1}^{n} n_k\gamma_k$

<u>DEFINE</u> $\displaystyle W(\gamma, p) = \sum_{k=1}^{n} n_k W(\gamma_k, p)$

<u>THM</u>: FOR ANY CHAINS γ, η

(i) $W(-\gamma, p) = -W(\gamma, p)$ $\quad p \in \mathbb{C} \setminus \{\gamma\}$

(ii) $W(\gamma + \eta, p) = W(\gamma, p) + W(\eta, p)$ $\quad p \in \mathbb{C} \setminus (\{\gamma\} \cup \{\eta\})$

(iii) IF γ IS A CYCLE,

$W(\gamma; \cdot) \in \mathbb{Z}$ AND CONSTANT ON COMPONENTS OF $\mathbb{C} \setminus \{\gamma\}$, VANISHING ON THE UNBDD. COMP.

Pf/ LET γ BE A CYCLE, SO $\gamma = \sum_{k=1}^{m} n_k \gamma_k$.

(i) AND (ii) ARE IMMEDIATE.

FOR (iii), DECOMPOSE $\gamma = \sum n_k \gamma_k$ INTO CYCLES $\gamma = \sum_{j=1}^{N} \eta_j$

SO
$$W(\gamma, p) = \sum_{k=1}^{m} n_k W(\gamma_k, p) = \sum_{j=1}^{N} W(\eta_j, p)$$

SINCE EACH η_j IS CLOSED, THE SUM IS AN INTEGER.

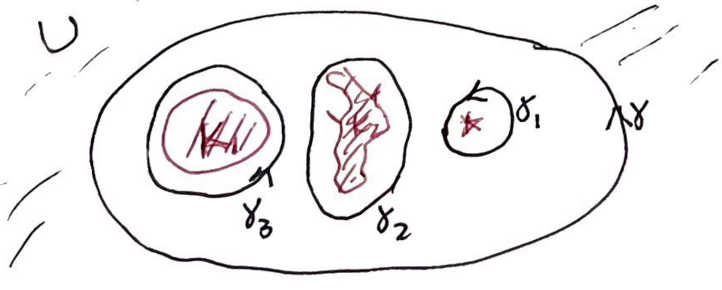
DEF LET $U \subset \mathbb{C}$ BE OPEN AND NON-EMPTY.

- A CYCLE $\gamma \subset U$ IS <u>NULL HOMOLOGOUS IN U</u> IF, FOR EVERY $p \in \mathbb{C} \setminus U$, $W(\gamma, p) = 0$.
  WRITE $\gamma \sim 0$
- TWO CYCLES γ AND η ARE HOMOLOGOUS $(\gamma \sim \eta)$ IF THEIR DIFFERENCE $\gamma - \eta$ IS NULL HOMOLOGOUS $(\gamma - \eta) \sim 0$
  IE $W(\gamma, p) = W(\eta, p)$

THE RELATION $\sim$ IS AN EQUIVALENCE RELATION ON THE GROUP OF CYCLES. THE EQUIVALENCE CLASS OF A CYCLE $\sim$ IS THE <u>HOMOLOGY CLASS OF γ</u> DENOTED $\langle \gamma \rangle$

THE SET OF ALL HOMOLOGY CLASSES ON U FORM AN ABELIAN GROUP (THE FIRST HOMOLOGY GROUP $H_1(U)$)

IE $H_1(U) = \{\text{CYCLES IN } U\} / \{\gamma \mid \gamma \sim 0 \text{ IN } U\}$.



$$\gamma \sim \gamma_1 + \gamma_2 + \gamma_3$$

- SPOZE THE CONCATENATION OF n CURVES IN U IS CLOSED, THEN $\gamma_1 \cdot \gamma_2 \cdot \gamma_3 \cdots \gamma_n \sim \gamma_1 + \gamma_2 + \cdots + \gamma_n$
- EVERY CYCLE IN U IS HOMOLOGOUS TO A FINITE SUM OF CLOSED CURVES IN U.

THM: (HOMOTOPIC $\Rightarrow$ HOMOLOGOUS) LET γ, η BE CURVES IN AN OPEN SET $U \subset \mathbb{C}$. AND $p \in U$

- IF $\gamma \simeq \eta$ IN (U, p), THEN $\gamma \sim \eta$
- IF γ AND η ARE FREELY HOMOTOPIC IN U, THEN $\gamma \sim \eta$

THE CONVERSE IS FALSE (IE $\gamma \sim \eta \not\Rightarrow \gamma \simeq \eta$.)

PROOF: (i) TAKE $p \in \mathbb{C} \setminus U$. γ, η HOMOTOPIC IN $\mathbb{C} \setminus \{p\}$ SO $W(\gamma - \eta, p) = W(\gamma, p) - W(\eta, p) = 0$ SO $\gamma - \eta \sim 0$
(ii) IF FREE HOMOTOPIC IN $\mathbb{C} \setminus U$, THEN ALSO IN $\mathbb{C} \setminus \{p\}$ FOR ANY $p \in \mathbb{C} \setminus U$. THUS WINDING # SAME

~~THE MAP~~ LET U BE A DOMAIN AND FIX $p \in U$

THEN THE MAP $\varphi : \pi_1(U, p) \to H_1(U)$ WITH

$$\varphi([\gamma]) = \langle \gamma \rangle$$

IS WELL DEFINED, AND IS A GROUP HOMOMORPHISM.

THE KERNEL OF φ ~~[illegible]~~ IS THE COMMUTATOR SUBGROUP $C \in \pi_1(U, p)$ GENERATED BY ELEMENTS OF THE FORM $[\gamma] \cdot [\eta] \cdot [\gamma]^{-1} \cdot [\eta]^{-1}$

THIS YIELDS AN ISOMORPHISM

$$\pi_1(U, p)/C \to H_1(U).$$

PROOF: LET γ BE A CYCLE, DECOMPOSE INTO CLOSED CURVES SO $\gamma \sim \sum \eta_k$. BUT EACH η_k IS NULL HOMOTOPIC.

THM: LET $U \subset \mathbb{C}$ BE SIMPLY CONNECTED. THEN $H_1(U) = 0$

THE CONVERSE IS ALSO TRUE.

NOW, WE WANT TO STATE CAUCHY'S THM IN TERMS OF HOMOLOGY, TO GIVE IT A MORE GENERAL SETTING.

TO DO SO, WE NEED TO DEFINE WHAT INTEGRATION OVER A CHAIN $\gamma = \sum_{j=1}^{m} n_j \gamma_j$ SHOULD BE. (CAN ASSUME EACH γ_j PIECEWISE C')

IF $f : \{\gamma\} \to \mathbb{C}$ IS CONTINUOUS, THEN

$$\int_\gamma f(z)\,dz = \sum_{k=1}^{m} n_k \int_{\gamma_k} f(z)\,dz$$

$$\text{SO } \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-p} = \sum_{k=1}^{m} n_k W(\gamma_k, p) = W(\gamma, p).$$

THM: LET γ BE A PIECEWISE C' CYCLE IN AN OPEN SET $U \subset \mathbb{C}$

IF $\gamma \sim 0$ AND $f \in \mathcal{O}(U)$ THEN

$$\int_\gamma f(z)\,dz = 0 \quad \text{AND} \quad f(z)\cdot W(\gamma, z) = \frac{1}{2\pi i}\int \frac{f(\zeta)}{\zeta - z}\,d\zeta \quad \text{FOR } z \in U \setminus \{\gamma\}.$$

COR IF γ, η ARE HOMOLOGOUS CYCLES (OR P.W. C' HOMOTOPIC) IN U AND $f \in \mathcal{O}(U)$ THEN

$$\int_\gamma f(z)\,dz = \int_\eta f(z)\,dz$$

PF/ APPLY THM TO $(\gamma - \eta)$ WHICH IS NULL HOMOLOGOUS

COR: LET $U \subset \mathbb{C}$ BE SIMPLY CONNECTED DOMAIN, $f \in \mathcal{O}(U)$. THEN $\int_\gamma f(z)\,dz = 0$ AND $f(z)\cdot W(\gamma, z) = \frac{1}{2\pi i}\int \frac{f(\zeta)}{\zeta - z}\,d\zeta$ FOR ANY CYCLE IN U

PF OF THM/ FIRST, ASSUME $\boxed{(*)\quad f(z)\, W(\gamma,z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta}$

FOR $f \in \mathcal{O}(U)$. THEN, FOR $p \in U \setminus \{\gamma\}$, LET $F(z) = (z-p) f(z)$

$$\int_\gamma f(z)\,dz = \int_\gamma \frac{F(z)}{z-p}\,dz = 2\pi i\, F(p) \cdot W(\gamma,p) = 0$$

PROVING THE FIRST PART.

SO TO ESTABLISH $(*)$, NOTE THIS IS EQUIVALENT TO

$$(**) \qquad \int_\gamma \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta = 0 \quad \text{FOR } z \in U \setminus \{\gamma\}$$

AND CONSIDER

$$g(\zeta, z) = \begin{cases} \dfrac{f(\zeta) - f(z)}{\zeta - z} & \text{FOR } z \neq \zeta \\ f'(z) & \text{FOR } z = \zeta \end{cases}$$

WHICH IS CONTINUOUS IN $U \times U$.

FURTHER, FOR FIXED ζ, $z \mapsto g(\zeta, z)$ IS HOLOMORPHIC IN U (SINCE HOLO IN $U \setminus \{\zeta\}$ AND CONTINUOUS).

THUS

$$G(z) = \frac{1}{2\pi i}\int_\gamma g(\zeta, z)\, d\zeta$$

HAS $G \in \mathcal{O}(U)$. IF WE SHOW $G(z) = 0$ FOR $z \in U$, $(**)$ FOLLOWS, AND HENCE $(*)$ AND THE THEOREM.

LET V BE THE OPEN SET OF $z \in \mathbb{C} \setminus \{\gamma\}$ FOR WHICH $W(\gamma, z) = 0$. SINCE $\gamma \sim 0$ IN U, V CONTAINS $\mathbb{C} \setminus U$, IE $U \cup V = \mathbb{C}$

NOW LET $\hat{G}(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$ FOR $z \in V$.

$\hat{G}$ IS HOLOMORPHIC ON V (WE SHOWED THIS ON 2/1, P11 OF NOTES)

IF $z \in U \cap V$, THEN $G(z) = \hat{G}(z)$ SINCE

$$G(z) = \left(\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{z - \zeta}\, d\zeta\right) - f(z)\, W(\gamma, z) = \hat{G}(z)$$

So let $\varphi = \begin{cases} G(z) & z \in U \\ \hat{G}(z) & z \in V \end{cases}$. φ is entire.

Choose R large enough that $\mathbb{D}_R(0)$ contains $\{\gamma\}$.
Observe ~~Since~~ $W(\gamma, z)$ is 0 on unbounded component of $\mathbb{C} \setminus \{\gamma\}$;
in particular, $\{z \mid |z| > R\} \subset V$.
Taking $\zeta \in \{\gamma\}$ and $|z| > 2R \Rightarrow |\zeta - z| > R$, so
by the ML inequality,

$$|\hat{G}(z)| \leq \frac{1}{2\pi R} \sup_{\zeta \in \{\gamma\}} |f(\zeta)| \cdot \text{length}(\gamma)$$

for $|z| > 2R$. But then

$$\lim_{z \to \infty} \varphi(z) = \lim_{z \to \infty} \hat{G}(z) = 0$$

and so by Liouville's thm, $\varphi(z) \equiv Z$
and hence $G = 0$ in U.

Combining the above results gives

$$\boxed{\textbf{Thm} \quad \text{Every holomorphic function in a simply connected domain has a primitive}}$$

Cor: Let $f \in \mathcal{O}(U)$ with γ a piecewise C^1 null homologous cycle in U. Then for $n \in \mathbb{Z}^+$

$$\boxed{f^{(n)}(z) \cdot W(\gamma, z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta \qquad \text{for } z \in U \setminus \{\gamma\}}$$

Finally, we can drop the C^1 restriction on γ:
every curve $\eta : [0,1] \to U$ is homotopic to one which is piecewise C^1.

SINGULARITIES

DEF: LET $p \in U$, WITH $U \subset \mathbb{C}$ OPEN, AND $f \in \mathcal{O}(U \setminus \{p\})$.

THEN p IS AN ISOLATED SINGULARITY OF f.

p IS A REMOVABLE SINGULARITY IF f CAN BE EXTENDED TO $g \in \mathcal{O}(U)$, IE

$$f(z) = g(z) \quad \text{FOR } z \in U \setminus \{p\}$$

EXAMPLES: • $f(z)$ A RATIONAL FUNCTION $\frac{p(z)}{q(z)}$, WITH p, q POLYNOMIALS. IF $q(z_0) = 0$ BUT $p(z_0) \neq 0$, THEN z_0 IS NOT REMOVABLE $\left(\lim_{z \to z_0} f(z) = \infty \right)$.

IF $p(z_0) = 0$ AND $q(z_0) = 0$, THEN z_0 IS REMOVABLE.

ALSO $\frac{e^z - 1}{z}$ HAS A REMOVABLE SINGULARITY AT $z = 0$, SINCE

$$e^z - 1 = \sum_{n=1}^{\infty} \frac{z^n}{n!} = z\underbrace{\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + \dots\right)}_{\text{ENTIRE.}}$$

• $f(z) = \frac{1}{\sin(1/z)}$ HAS AN ISOLATED SINGULARITY AT $\frac{1}{n\pi}$ FOR $n \in \mathbb{Z} \setminus \{0\}$.

BUT 0 IS NOT AN ISOLATED SINGULARITY.

BY MORERA'S THM (ON 2/1) IF p IS AN ISOLATED SINGULARITY OF f WITH $\lim_{z \to p} f(z)$ EXISTING, p IS REMOVABLE.

A WEAKER HYPOTHESIS BY RIEMANN:

THM RIEMANN'S REMOVABILITY THM (1851)

> IF A HOLOMORPHIC FUNCTION IS BOUNDED IN A NEIGHBORHOOD OF AN ISOLATED SINGULARITY, IT IS REMOVABLE

Pf/ LET f BE BOUNDED & HOLOMORPHIC IN THE PUNCTURED DISK $\mathbb{D}^*_r(p)$.

THEN $g(z) = (z-p)^2 f(z)$ IS HOLOMORPHIC IN $\mathbb{D}^*_r(p)$ WITH $\lim\limits_{z \to p} g(z) = 0$. THUS WE CAN EXTEND g TO $\mathbb{D}_r(p)$ BY SETTING $g(p) = 0$.

BY THE BOUNDEDNESS OF f,

$$\lim_{z \to p} \frac{g(z) - g(p)}{z - p} = \lim_{z \to p} (z-p) f(z) = 0$$

SO $g'(p)$ EXISTS WITH $g'(p) = 0$.

IN $\mathbb{D}_r(p)$ WE HAVE $g(z) = \sum\limits_{n=2}^{\infty} a_n (z-p)^n$.

THUS $\sum\limits_{n=2}^{\infty} a_n (z-p)^{n-2}$ IS A HOLO. EXTENSION OF f TO $\mathbb{D}_r(p)$ ☒

NOTE:

WE DON'T REALLY NEED f BOUNDED, JUST THAT $\lim\limits_{z \to p} (z-p) f(z) = 0$.

FOR EXAMPLE, IF $|f(z)| < C|z|^{\alpha}$ FOR $C \in \mathbb{C}$, $\alpha \in (0,1)$,

THEN 0 IS REMOVABLE